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Some spectral invariants of the neighborhood corona of graphs

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ABSTRACT

Given two graphs, G_1 with vertices $\{v_1, v_2, \ldots, v_n\}$ and G_2, the *neighborhood corona*, $G_1 \star G_2$, is the graph obtained by taking n copies of G_2 and joining by an edge each neighbor of v_i, in G_1, to every vertex of the ith copy of G_2. A special instance $G_1 \star K_1$ of the neighborhood corona is called the splitting graph of G_1 and has a property that its spectrum consists of all eigenvalues $\phi\lambda$ and $-\phi^{-1}\lambda$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio and λ is an arbitrary eigenvalue of G_1. In this paper, various spectra invariants of the neighborhood corona of graphs are studied. First, the condition number, the inertia, and the HOMO–LUMO gap of the s-fold splitting graphs are investigated, some of which turn out to have the golden-ratio scaling with the corresponding invariants of the original graph. Then, resistance distances and the Kirchhoff index of the neighborhood corona graph $G_1 \star G_2$ are computed, with explicit expressions being obtained, which extends the previously known result.

1. Introduction

In this paper, we consider spectral invariants of neighborhood corona of graphs. Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Suppose that $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$. Then the *neighborhood corona*, $G_1 \star G_2$, is the graph obtained by taking n_1 copies of G_2 and joining by an edge each neighbor of v_i, in G_1, to every vertex of the ith copy of G_2 [6]. Suppose further that $|V(G_2)| = n_2$, $|E(G_1)| = m_1$ and $|E(G_2)| = m_2$. Then it is easily verified that the neighborhood corona $G_1 \star G_2$, of G_1 and G_2, has $n_1 + n_1 n_2$ vertices and $(2n_2 + 1)m_1 + n_1 m_2$ edges [6]. When $G_2 = K_1$, i.e., G_2 is an isolated vertex, then $G_1 \star G_2$ is called the *splitting graph* of G_1 as defined in [13]. For example, the neighborhood corona of $C_4 \star K_2$ and the splitting graph of C_4 are shown in Fig. 1. Note that in general this operation is not commutative.

In the following, we introduce spectral invariants under consideration. We first introduce some spectral invariants of the adjacency matrix. Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of G, denoted by $A(G)$, is a $n \times n$ matrix whose ij-th entry is 1 if v_i and v_j are adjacent and 0 otherwise. The *spectrum* $\mathrm{Sp}(G) = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$ of G is the family of all eigenvalues of the adjacency matrix $A(G)$ of G. If $A(G)$ is nonsingular, then we can consider condition numbers of $A(G)$. Actually, condition numbers are defined for more general nonsingular symmetric matrices. We call a nonsingular symmetric matrix A to be "well conditioned" if the solution of the system of linear equations $Ax = b$ is stable, *i.e.*, if it varies little with a small change in the elements of matrix A and in the column b of the free terms. Some quantitative features of conditionality are known in literature—the so-called *condition numbers*. One of such numbers is $\mathrm{Cond}(A) = \lambda_1/|\lambda_{|\min|}|$, where λ_1 is the maximum (positive) eigenvalue, and $\lambda_{|\min|}$ is the minimum-modulus eigenvalue, of A, respectively. The smaller $\mathrm{Cond}(A)$ is, the better it is for solving a respective system of linear equations; and conversely, the greater is this

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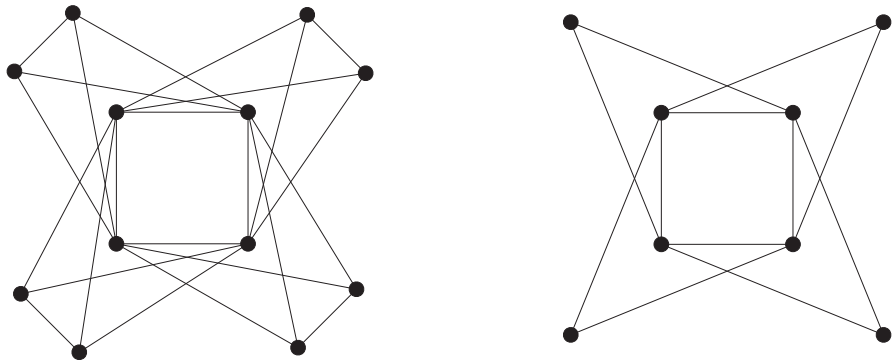

Fig. 1. The neighborhood corona $C_4 \star K_2$ (left) and the splitting graphs of C_4 (right).

quantity, the less stable may be the solution. Another spectral invariant is the *inertia* of G, which is defined to be the triple $In(G) = \left(i_+(G), i_-(G), i_0(G)\right)$, where $i_+(G)$, $i_-(G)$, $i_0(G)$ are the numbers of the positive, negative, and zero eigenvalues of $A(G)$ including multiplicities, respectively. Traditionally, the number $i_0(G)$ is called the *nullity* of G. $i_+(G)$ and $i_-(G)$ are called the *positive, negative index* of inertia, respectively. The last invariant is the *HOMO–LUMO gap (or HOMO–LUMO separation)* $\Delta(G)$ of G, which is defined after the Hückel method of molecular orbitals, as the difference $\lambda_H - \lambda_L$, where $H = \lfloor \frac{n+1}{2} \rfloor$ and $L = \lceil \frac{n+1}{2} \rceil$. λ_H and λ_L are called *median eigenvalues* of G. Many physicochemical parameters of molecules are determined by or are dependent upon the HOMO–LUMO gap. The HOMO–LUMO gap is very important in quantum physics, because it determines the conductivity of a substance. Usually, the smaller $\Delta(G)$ is, the greater is the conductivity.

Now we turn to the spectral invariants of the Laplacian matrix of G. Recall that the *Laplacian matrix* $L(G)$, of G, is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees. For a connected graph G, it is well known that $L(G)$ is positive semidefinite, and that it has exactly one zero eigenvalue, and all the other eigenvalues are positive. Here, we consider the resistance distance and the Kirchhoff index of graphs, whose definitions seem to be far apart from the Laplacian spectra, but which turn out to be spectral invariants of the Laplacian matrix. The *resistance distance* [9] between any two vertices v_i and v_j of G, denoted by $\Omega_G(v_i, v_j)$, is defined to be the net effective resistance between them in the electrical network, constructed from G by replacing each edge with a unit resistor. The *Kirchhoff index* (or *effective resistance*) of G [9], denoted by $Kf(G)$, is defined as the sum of *resistance distances* between all pairs of vertices, *i.e.*,

$$Kf(G) = \sum_{i<j} \Omega_G(v_i, v_j).$$

The resistance distance and the Kirchhoff index are spectral invariants of the Laplacian matrix that can be computed as follows:

Theorem 1 ([8]). *Let G be a connected graph of order n with Laplacian eigenvalues $0 = \theta_1, \theta_2, \ldots, \theta_n$. Suppose $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are corresponding normalized orthogonal eigenvectors. Then,*

$$\Omega_G(i, j) = \sum_{k=2}^{n} \frac{(x_{ki} - x_{kj})^2}{\theta_k}, \tag{1}$$

where x_{ki} and x_{kj} are the ith and jth components of $\mathbf{X}_k$, respectively.

Theorem 2 ([21,7]). *Let G be a connected graph of order n with Laplacian eigenvalues $0 = \theta_1, \theta_2, \ldots, \theta_n$. Then, the Kirchhoff index of G can be computed as*

$$Kf(G) = n \sum_{i=2}^{n} \frac{1}{\theta_i}. \tag{2}$$

Actually, the study of resistance distance could be dated back to the classical work of Kirchhoff. In addition, owing to the work of Klein and Randić in 1993 [9], people notify that the effective resistance is a distance function defined on graphs. From then on, resistance distance and the Kirchhoff index have been widely studied in mathematics, physics and chemistry literature. For more information, the readers are referred to [18–20,11,17,16,14,2] and references therein.

The rest of this paper is organized as follows. In Section 2, the spectral and Laplacian spectral of the neighborhood corona of graphs are given, with some preliminary consequences being presented. Section 3 describes the main results of the paper, which consists of two subsections. In Section 3.1, the condition number, the inertia, and the HOMO–LUMO gap of the s-fold splitting graphs are investigated, some of which turn out to have a golden-ratio scaling with the invariants of the original