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Minimum multiplicity edge coloring via orientation

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ABSTRACT

We study an edge coloring problem in multigraphs, in which each node incurs a cost equal to the number of appearances of the most frequent color among those received by its incident edges. We seek an edge coloring with a given number w of colors, that minimizes the total cost incurred by the nodes of the multigraph. We consider a class of approximation algorithms for this problem, which are based on orienting the edges of the multigraph, then grouping appropriately the incoming and outgoing edges at each node so as to construct a bipartite multigraph of maximum degree w, and finally obtaining a proper edge coloring of this bipartite multigraph. As shown by Nomikos et al. (2001), simply choosing an arbitrary edge orientation in the first step yields a 2-approximation algorithm. We investigate whether this approximation ratio can be improved by a more careful choice of the edge orientation in the first step. We prove that, assuming a worst-case bipartite edge coloring, this is not possible in the asymptotic sense, as there exists a family of instances in which any orientation gives a solution with cost at least $2 - \Theta\left(\frac{1}{w}\right)$ times the optimal. On the positive side, we show how to produce an orientation which results in a solution with cost within a factor of $2 - \frac{1}{2^w}$ of the optimal, thus yielding an approximation ratio strictly better than 2. This improvement is important in view of the fact that this graph-theoretic problem models, among others, wavelength assignment to communication requests in multifiber optical star networks. In this context, the parameter w corresponds to the number of available wavelengths per fiber, which is limited in practice due to technological constraints.

1. Introduction

Let $G = (V, E)$ be an undirected multigraph without self-loops. Given a coloring of its edges, let $\mu(v, c)$ denote the number of edges incident to v that have received color c and let $\mu(v) = \max_c \mu(v, c)$. We will call $\mu(v, c)$ the *multiplicity of* c *at* v and $\mu(v)$ the *multiplicity of* v. In the Minimum Multiplicity Edge Multicoloring problem (MinMult-EMC), one seeks an edge coloring with a given number of colors, that minimizes the sum of node multiplicities. Formally:

Problem 1 (MinMult-EMC).
Instance: $\langle G, w\rangle$, where $G = (V, E)$ is an undirected multigraph and $w \in \mathbb{N}$ is the number of available colors.
Feasible solution: a coloring of E with w colors.
Goal: minimize $\sum_{v \in V} \mu(v)$.

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Algorithm 1 A 2-approximation algorithm for MinMult-EMC [18]

Input: an instance $\langle G, w\rangle$ of MinMult-EMC, $G = (V, E)$
Output: a 2-approximate solution

1: Assign an arbitrary direction to each edge of G.
2: For each $v \in V$, group its d_v^{out} outgoing edges into $\left\lceil \frac{d_v^{\text{out}}}{w} \right\rceil$ groups of at most w edges each, and let V_{out} denote the set of all groups of outgoing edges. Similarly, for each $v \in V$, group its d_v^{in} incoming edges into $\left\lceil \frac{d_v^{\text{in}}}{w} \right\rceil$ groups of at most w edges each, and let V_{in} denote the set of all groups of incoming edges.
3: Construct the bipartite multigraph $H = (V_{\text{out}} \cup V_{\text{in}}, A)$, where for each edge in E, A contains one edge joining its outgoing group to its incoming group. The maximum degree of H is bounded by w.
4: Compute a proper edge coloring of H with w colors.
5: Assign to each edge of G the color of the corresponding edge in H.

There is a large literature on edge coloring, which typically considers the problem from the point of view of minimizing the number of colors used, under various constraints imposed on the obtained coloring. To the best of our knowledge, the MinMult-EMC problem, which has a different objective function, has not been studied as such in the literature. However, in view of the diverse applications of edge coloring in domains such as job scheduling, routing, network resource allocation, etc. [4,8,12,13,16], it is not surprising that MinMult-EMC appears and has, in fact, been considered implicitly in the context of wavelength allocation in multifiber optical networks [18].

We recall some known results and we make some preliminary observations on MinMult-EMC in Section 1.1.

Notation. Throughout the paper, d_v will denote the degree of a node v in an undirected multigraph, whereas for directed multigraphs we will use d_v^{in} (resp. d_v^{out}) for the in-degree (resp. out-degree) of node v. An *orientation* of an undirected multigraph is a directed multigraph in which each edge $\{u, v\}$ is replaced by one of the arcs (u, v) or (v, u). If G is a graph or a multigraph, $V(G)$ is the node set of G and $E(G)$ is the edge set of G. For $k \geq 2$, C_k denotes the undirected cycle of size k and K_k denotes the clique of size k. We use the binary operation $a \bmod b$ for positive integers a, b, which gives the remainder of the division a/b. If A is an event in a suitable sample space, then $\mathbb{P}[A]$ denotes the probability of A.

1.1. Preliminaries

Fact 1. *Under any edge coloring with w colors, the multiplicity of each node v is at least $\left\lceil \frac{d_v}{w} \right\rceil$, thus the minimum cost for any* MinMult-EMC *instance is at least $\sum_{v\in V} \left\lceil \frac{d_v}{w} \right\rceil$.*

For any fixed $w \geq 3$, MinMult-EMC is NP-hard via a straightforward reduction from the decision version of the classical edge coloring problem on w-regular graphs, which is known to be NP-complete [11,14]. Nomikos et al. [18] propose a 2-approximation algorithm which we restate as Algorithm 1 in MinMult-EMC terms (the algorithm was originally stated in terms of wavelength allocation in multifiber optical networks). The analysis in [18] is tight, as there exists a family of instances in which Algorithm 1 computes a solution with cost exactly twice the optimum: $\{\langle C_k, w\rangle : \text{even } k \geq 2 \text{ and } w \geq 2\}$. Indeed, if the directions assigned in step 1 are such that each node has in-degree 1 and out-degree 1, then the resulting bipartite multigraph H will contain k edges that can all be colored with the same color. Translated to the original instance, this induces a cost of 2 for each node for a total cost of $2k$, whereas the optimum solution has cost k by coloring the edges with alternating colors around the cycle.

Definition 1. Let $\langle G, w\rangle$ be an instance of MinMult-EMC and fix an orientation of G. We say that a node v is *locally optimal* if the following condition holds:

$$\left(d_v^{\text{in}} \bmod w = 0\right) \vee \left(d_v^{\text{out}} \bmod w = 0\right) \vee \left((d_v^{\text{in}} \bmod w) + (d_v^{\text{out}} \bmod w) > w\right)$$

The pertinence of locally optimal nodes is revealed by the following lemma, which is implicit in the analysis in [18].

Lemma 2 (*[18]*)**.** *In any solution computed by Algorithm 1, each node v incurs a cost of exactly $\left\lceil \frac{d_v}{w} \right\rceil$ if it is locally optimal with respect to the directions assigned during step 1, or at most $\left\lceil \frac{d_v}{w} \right\rceil + 1$ if it is not locally optimal.*

In other words, Algorithm 1 incurs an additional cost, with respect to the lower bound of Fact 1, of at most one for each non-locally-optimal node. In fact, as we prove in Section 2 (Lemma 3), for every orientation of the given graph and for every edge grouping that can be chosen in steps 1 and 2 of Algorithm 1, there exists a worst-case proper edge coloring of the resulting bipartite multigraph (step 4 of Algorithm 1) that causes *every* non-locally-optimal node v to contribute a cost of exactly $\left\lceil \frac{d_v}{w} \right\rceil + 1$.

If $w = 2$, then the problem can be solved exactly in polynomial time: The Euler partition algorithm in [7] computes a partition of the edges of a multigraph into open and closed paths, with the property that each vertex of odd degree is the extremity of exactly one open path, and each vertex of even degree is the extremity of no open paths. Note, then, that if