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Note

On the precise value of the strong chromatic index of a planar graph with a large girth

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A B S T R A C T

A *strong k-edge-coloring* of a graph G is a mapping from $E(G)$ to $\{1, 2, \ldots, k\}$ such that every pair of distinct edges at distance at most two receive different colors. The *strong chromatic index* $\chi_s'(G)$ of a graph G is the minimum k for which G has a strong k-edge-coloring. Denote $\sigma(G) = \max_{xy\in E(G)}\{\deg(x) + \deg(y) - 1\}$. It is easy to see that $\sigma(G) \le \chi_s'(G)$ for any graph G, and the equality holds when G is a tree. For a planar graph G of maximum degree Δ, it was proved that $\chi_s'(G) \le 4\Delta + 4$ by using the Four Color Theorem. The upper bound was then reduced to 4Δ, $3\Delta + 5$, $3\Delta + 1$, 3Δ, $2\Delta - 1$ under different conditions for Δ and the girth. In this paper, we prove that if the girth of a planar graph G is large enough and $\sigma(G) \ge \Delta(G)+2$, then the strong chromatic index of G is precisely $\sigma(G)$. This result reflects the intuition that a planar graph with a large girth locally looks like a tree.

1. Introduction

A *strong k-edge-coloring* of a graph G is a mapping from $E(G)$ to $\{1, 2, \ldots, k\}$ such that every two distinct edges at distance at most two receive different colors. It induces a proper vertex coloring of $L(G)^2$, the square of the line graph of G. The *strong chromatic index* $\chi_s'(G)$ of G is the minimum k for which G has a strong k-edge-coloring. This concept was introduced by Fouquet and Jolivet [18,19] to model the channel assignment in some radio networks. For more applications, see [4,29,32,31,23,36].

A Vizing-type problem was asked by Erdős and Nešetřil, and further strengthened by Faudree, Schelp, Gyárfás and Tuza to give an upper bound for $\chi_s'(G)$ in terms of the maximum degree $\Delta = \Delta(G)$:

Conjecture 1 (*Erdős and Nešetřil '88 [15] '89 [16], Faudree et al. '90 [17]*). *If G is a graph with maximum degree Δ, then $\chi_s'(G) \le \Delta^2 + \lfloor \frac{\Delta}{2} \rfloor^2$.*

As demonstrated in [17], there are indeed some graphs that reach the conjectured upper bound.

By a greedy algorithm, it can be easily seen that $\chi_s'(G) \le 2\Delta(\Delta - 1) + 1$. Molloy and Reed [27] used the probabilistic method to show that $\chi_s'(G) \le 1.998\Delta^2$ for maximum degree Δ large enough. Recently, this upper bound was improved by Bruhn and Joos [8] to $1.93\Delta^2$.

For small maximum degrees, the cases $\Delta = 3$ and 4 were studied. Andersen [1] and Horák et al. [21] proved that $\chi_s'(G) \le 10$ for $\Delta(G) \le 3$ independently; and Cranston [12] showed that $\chi_s'(G) \le 22$ when $\Delta(G) \le 4$.

According to the examples in [17], the bound is tight for $\Delta = 3$, and the best we may expect for $\Delta = 4$ is 20.

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The strong chromatic index of a few families of graphs are examined, such as cycles, trees, d-dimensional cubes, chordal graphs, Kneser graphs, k-degenerate graphs, chordless graphs and C_4-free graphs, see [5,11,14,17,26,39,41]. As for Halin graphs, refer to [10,24,25,34,35]. For the relation to various graph products, see [37].

Now we turn to planar graphs.

Faudree et al. used the Four Color Theorem [2,3] to prove that planar graphs with maximum degree Δ are strongly $(4\Delta+4)$-edge-colorable [17]. The conclusion can be extended to K_5-minor free graphs [28]. Moreover, every planar G with girth at least 7 and $\Delta \geq 7$ is strongly 3Δ-edge-colorable by applying a strengthened version of Vizing's Theorem on planar graphs [33,38] and Grőtzsch's theorem [20].

The following results are obtained by using a discharging method:

Theorem 2 (*Hudák et al. '14 [22]*). *If G is a planar graph with girth at least 6 and maximum degree at least 4, then $\chi_s'(G) \leq 3\Delta(G) + 5$.*

Theorem 3 (*Hudák et al. '14 [22]*). *If G is a planar graph with girth at least 7, then $\chi_s'(G) \leq 3\Delta(G)$.*

And the bounds are improved by Bensmail et al.

Theorem 4 (*Bensmail et al. '14 [6]*). *If G is a planar graph with girth at least 6, then $\chi_s'(G) \leq 3\Delta(G) + 1$.*

Theorem 5 (*Bensmail et al. '14 [6]*). *If G is a planar graph with girth at least 5 or maximum degree at least 7, then $\chi_s'(G) \leq 4\Delta(G)$.*

It is also interesting to see the asymptotic behavior of strong chromatic index when the girth is large enough.

Theorem 6 (*Borodin and Ivanova '13 [7]*). *If G is a planar graph with maximum degree $\Delta \geq 3$ and girth at least $40\lfloor \frac{\Delta}{2} \rfloor + 1$, then $\chi_s'(G) \leq 2\Delta - 1$.*

Theorem 7 (*Montassier, Pêcher and Raspaud '13 [28]*). *If G is a planar graph with maximum degree $\Delta \geq 4$ and girth at least $10\Delta + 46$, then $\chi_s'(G) \leq 2\Delta - 1$.*

Theorem 8 (*Wang and Zhao '15 [40]*). *If G is a planar graph with maximum degree $\Delta \geq 4$ and girth at least $10\Delta - 4$, then $\chi_s'(G) \leq 2\Delta - 1$.*

The concept of maximum average degree is also an indicator to the sparsity of a graph. A folklore lemma, which can be proved by Euler's formula, points out the relation between the girth of a planar graph and its maximum average degree.

Lemma 9. *A planar graph G with girth g has the maximum average degree $\mathrm{mad}(G) < 2 + \frac{4}{g-2}$.*

Many results concerning planar graphs with large girths can be extended to general graphs with small maximum average degrees and large girths. Strong chromatic index is no exception.

Theorem 10 (*Wang and Zhao '15 [40]*). *Let G be a graph with maximum degree $\Delta \geq 4$. If $\mathrm{mad}(G) < 2 + \frac{1}{3\Delta-2}$, the even girth is at least 6 and the odd girth is at least $2\Delta - 1$, then $\chi_s'(G) \leq 2\Delta - 1$.*

In terms of maximum degree Δ, the bound $2\Delta - 1$ is best possible. We seek for a better parameter as a refinement. Define

$$\sigma(G) := \max_{xy \in E(G)} \{\deg(x) + \deg(y) - 1\}.$$

An *antimatching* is an edge set $S \subseteq E(G)$ in which any two edges are at distance at most 2, thus any strong edge-coloring assigns distinct colors on S. Notice that each color set of a strong edge-coloring is an induced matching, and the intersection of an induced matching and an antimatching contains at most one edge. The fact suggests a dual problem to strong edge-coloring: finding a maximum antimatching of G, whose size is denoted by $\mathrm{am}(G)$. For any edge $xy \in E(G)$, the edges adjacent to xy form an antimatching of size $\deg(x) + \deg(y) - 1$. Together with the weak duality, this gives the inequality

$$\chi_s'(G) \geq \mathrm{am}(G) \geq \sigma(G).$$

By induction, we see that for any nontrivial tree T, $\chi_s'(T) = \sigma(T)$ attains the lower bound [17]. Based on the intuition that a planar graph with large girth locally looks like a tree, in this paper, we focus on this class of graphs. More precisely, we prove the following main theorem:

Theorem 11 (*Main Theorem*). *If G is a planar graph with $\sigma = \sigma(G) \geq 5$, $\sigma \geq \Delta(G) + 2$ and girth at least $5\sigma + 16$, then $\chi_s'(G) = \sigma$.*

We also make refinement on the girth constraint and gain a stronger result in Section 4.

The condition $\sigma \geq \Delta(G) + 2$ is necessary as shown in the following example. Suppose $n \geq 1$ and $d \geq 2$. Construct $G_{3n+1,d}$ from the cycle $(x_1, x_2, \ldots, x_{3n+1})$ by adding $d - 2$ leaves adjacent to each x_{3i} for $1 \leq i \leq n$. Then $\sigma(G_{3n+1,d}) = d + 1 < d + 2 = \Delta(G_{3n+1,d}) + 2$. See Fig. 1 for $G_{3n+1,4}$.

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