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Safe number and integrity of graphs

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ABSTRACT

For a connected graph $G = (V(G), E(G))$, a non-empty subset S of $V(G)$ is a *safe set* if, for every component C of $G[S]$ and every component D of $G - S$, we have $|V(C)| \geq |V(D)|$ whenever there exists an edge of G between C and D. If $G[S]$ is connected, then S is called a *connected safe set*. The *safe number* $s(G)$ of G is defined as $s(G) := \min\{|S| : S \text{ is a safe set of } G\}$, and the *connected safe number* $cs(G)$ of G is defined as $cs(G) := \min\{|S| : S \text{ is a connected safe set of } G\}$. The integrity of a graph G is defined as $I(G) := \min\{|S| + \tau(G-S) : S \subseteq V(G)\}$, where $\tau(G - S)$ is the order of the largest component of $G - S$.

In this paper, we discuss a relationship between the (connected) safe number and the integrity in a connected graph.

1. Introduction

In much literature, several measures of the reliability of a communication network have been considered. In order to introduce a reasonable measure of network reliability, we consider some situations on a graph network model, where we have some possible attacks by outside enemies who want to destroy the graph network efficiently. Motivated by disrupting a graph network by removing a small set of vertices such that the remaining connected components are small, the integrity of a graph was introduced by Barefoot et al. [4]. This measure represents, in some sense, a trade-off between the amount of work done to damage the network and how badly the network is damaged. The formal definition of the integrity of a graph is given as follows.

For a graph H, let $\mathcal{C}(H)$ denote the family of components of H, and set $\tau(H) = \max\{|V(C)| : C \in \mathcal{C}(H)\}$. The *integrity* of a graph G is defined as $I(G) := \min\{|S| + \tau(G - S) : S \subseteq V(G)\}$. A subset S of $V(G)$ satisfying $|S| + \tau(G - S) = I(G)$ is called an *integrity set* of G. The integrity of some basic graph classes, such as paths, cycles and so on were determined by Bagga et al. [2], and Vince [9] showed that any cubic graph G of order n satisfies $I(G) < (n + 2\sqrt{6n} + 1)/3$. For other results, we also refer the reader to [8] by Goddard.

On the other hand, another new relevant measure, the *safe number* of a connected graph, was recently introduced by Fujita et al. [7].

Let G be a connected graph of order n. For a pair of vertex disjoint subsets M, N of $V(G)$, let $E_G(M, N)$ be the set of edges between M and N in G. A nonempty set $S \subseteq V(G)$ is a *safe set* of G if for all $C \in \mathcal{C}(G[S])$ and $D \in \mathcal{C}(G - S)$ with $E_G(V(C), V(D)) \neq \emptyset$, $|V(C)| \geq |V(D)|$. A safe set S of G is *connected* if $G[S]$ is connected. The minimum cardinality of a safe set (resp. a connected safe set) of G, denoted by $s(G)$ (resp. $cs(G)$), is the *safe number* (resp. the *connected safe number*) of G.

From the definition, it is easy to check the following.

Lemma 1.1 (*Fujita et al. [7]*). *Let G be a connected graph of order $n \geq 2$. Then $s(G) \leq cs(G) \leq \lceil \frac{n}{2} \rceil$.*

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It was shown in [7] that, for an integer p, the problem for asking whether $cs(G) \leq p$ is NP-complete. On the positive side, Fujita et al. [7] showed that this problem can be solved in linear time on trees. Also, Águeda et al. [1] showed that the same problem concerning $s(G)$ can be solved in $O(n^5)$ time on trees.

As a counterpart in the direction of the graph integrity, Clark et al. [5] showed that, for an integer p, the problem for asking whether $I(G) \leq p$ is NP-complete, even when restricted to planar graphs. Fellows and Stueckle [6] showed that the problem can be solved in $O(p^{3p}n)$ time, and is thus fixed-parameter tractable when parameterized by p.

In view of these results, we see that determining $cs(G)$, $s(G)$, $I(G)$ seems quite hard in general. In our attempt to determine these parameters, studying the magnitude relation among those parameters would be important, as it could give us a good insight to construct polynomial algorithms to determine these parameters in certain classes of graphs.

Motivated by this viewpoint, in this paper, we seek a relationship between the (connected) safe number and the integrity of a graph. We start with an upper bound on $I(G)$ in terms of the (connected) safe number. We now present the following proposition, which was originally shown in Bapat et al. [3] in a more generalized form. We give the proof for the convenience of readers.

Proposition 1.2 (*Bapat et al. [3]*)**.** *Let G be a connected graph. Then $I(G) \leq 2s(G)$ $(\leq 2cs(G))$.*

Proof. Let S be a safe set of G with $|S| = s(G)$, and let $D \in \mathcal{C}(G - S)$ with $|V(D)| = \tau(G - S)$. Since G is connected, there exists a vertex $x \in S$ such that $N_G(x) \cap V(D) \neq \emptyset$. Let C be the component of $G[S]$ containing x. Then $E_G(V(C), V(D)) \neq \emptyset$, and hence $|V(C)| \geq |V(D)|$. Consequently, we have

$$I(G) \leq |S| + \tau(G - S) = |S| + |V(D)| \leq |S| + |V(C)| \leq 2|S| = 2s(G),$$

as desired. □

Let K_m denote the *complete graph* of order n. For two graphs H_1 and H_2 and a positive integer l, let $H_1 + H_2$ and lH_1 denote the *join* of H_1 and H_2 and the disjoint union of l copies of H_1, respectively. Now we focus on the graph $K_m + lK_m$. It is clear that $I(K_m + lK_m) = 2m$ and $s(K_m + lK_m) = m$. In particular, $I(K_m + lK_m) = 2s(K_m + lK_m)$. Thus Proposition 1.2 is best possible.

Our main result in this paper is to give the tight lower bound on $I(G)$ in terms of the (connected) safe number.

Theorem 1.3. *Let G be a connected graph. Then*

(i) $I(G) \geq 2\sqrt{s(G) - 2} + 1$ *if G is not a star, and*
(ii) $I(G) \geq 2\sqrt{cs(G) - 1}$.

Remark 1. If a graph G is a star, then $s(G) = cs(G) = 1$. On the other hand, if a connected graph G satisfies $s(G) = 1$, then every component of $G - S$ consists of one vertex where S is a safe set of G with $|S| = 1$, and hence G is a star. Consequently, a connected graph G is not a star if and only if $s(G) \geq 2$. In particular, the value $s(G) - 2$ in Theorem 1.3(i) is a non-negative integer.

Let G be a connected graph. For $x, y \in V(G)$, let $d_G(x, y)$ denote the *distance* between x and y in G. For $x \in V(G)$, the integer $\mathrm{ecc}_G(x) := \max\{d_G(x, y) : y \in V(G)\}$ is called the *eccentricity* of x. The *radius* of G is the integer $\mathrm{rad}(G) := \min\{\mathrm{ecc}_G(x) : x \in V(G)\}$. A vertex x is a *center* of G if $\mathrm{ecc}_G(x) = \mathrm{rad}(G)$.

Our second main result claims that the ratio of the (connected) safe number and the integrity can be bounded by the radius of a connected graph.

Theorem 1.4. *Let G be a connected graph of order $n \geq 2$. Then $I(G) \geq \frac{cs(G)}{rad(G)}$ $\left(\geq \frac{s(G)}{rad(G)}\right)$.*

From our results, we know that, roughly speaking, the value $I(G)$ ranges from $2\sqrt{cs(G)}$ to $2s(G)$. It would be natural to ask what kind of graphs always satisfy $I(G) > cs(G)$ (or, $I(G) < cs(G)$, $I(G) > s(G)$, $I(G) = cs(G)$, ... and so on).

Highly connected graphs could be an answer to the above question.

Proposition 1.5. *Let G be a graph of order n. If G is $\lceil \frac{n}{2} \rceil$-connected, then $I(G) > cs(G)$ holds.*

Proof. Since any cut set of G has at least $\lceil \frac{n}{2} \rceil$ vertices, we have $I(G) > \lceil \frac{n}{2} \rceil$. Hence, in view of Lemma 1.1, it follows that $I(G) > cs(G)$. □

Remark 2. The complete bipartite graph $K_{\frac{n}{2}-1, \frac{n}{2}+1}$ shows that the connectivity condition in Proposition 1.5 is best possible, as $I(K_{\frac{n}{2}-1, \frac{n}{2}+1}) = cs(K_{\frac{n}{2}-1, \frac{n}{2}+1}) = \frac{n}{2}$.

The rest of this paper is organized as follows: In Section 2, we prove Theorem 1.3. In Section 3, we discuss the sharpness of Theorem 1.3. In Section 4, we prove Theorem 1.4.