Economics Letters 172 (2018) 19–22

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

# Axiomatic characterisations of the basic best–worst rule

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## HIGHLIGHTS

- We propose two new axiomatisations of the basic best–worst rule.
- We show that anonymity is not necessary to characterise the basic best–worst rule.
- Top–bottom non-negativity was used for the direct characterisation.

## ARTICLE INFO

*Article history:*
Received 19 June 2018
Received in revised form 10 August 2018
Accepted 11 August 2018
Available online xxxx

*JEL classification:*
D70
D71
D72

*Keywords:*
Best–worst
Anonymity
Top–bottom non-negativity

## ABSTRACT

We show that *reinforcement* and *top–bottom cancellation* imply *anonymity*, and that the basic best–worst rule can be characterised by *neutrality*, *continuity*, *reinforcement*, and *top–bottom cancellation*. Additionally, we directly characterise the basic best–worst rule by *neutrality*, *reinforcement*, *top–bottom non-negativity*, and *top–bottom cancellation*. *Top–bottom non-negativity* requires that if the difference in number between the individuals preferring a certain alternative as their best and worst alternatives respectively is strictly negative, then that alternative is not included in the social choice.



## 1. Introduction

The best–worst rules have made it easy for us to determine the social choice because we need to collect information only about each individual's best and worst alternatives. Some recent studies have analysed several applied models of the best–worst rules. For instance, Marley and Louviere (2005) analysed the probabilistic discrete choice models and Cahan and Slinko (2018) characterised multi-candidate pure-strategy equilibrium in the Hotelling–Downs spatial election model under the best–worst voting rules.

We have two types of best–worst rules in the literature. The first is the basic best–worst rule where we assign 1, $-1$, and 0 points for the best, worst, and other alternatives, respectively.[1] The second type is the weighted best–worst rule. Here, we assign $\alpha$, $-\beta$, and 0 points ($\alpha, \beta > 0$ and $\alpha \neq \beta$) for the best, worst, and other alternatives, respectively.[2] This study analyses the basic best–worst rule.

Young (1975) axiomatised a scoring social choice rule by *anonymity*, *neutrality*, *continuity*, and *reinforcement*.[3] This allows us to characterise specific scoring rules such as plurality, anti-plurality, best–worst, and the Borda rules using Young's (1975) theorem. In fact, Garciá-Lapresta et al. (2010) characterised the best–worst rules by using Young's (1975) theorem. However, we must check whether the axioms for characterisation can be relaxed, or whether there is an alternative approach.

The objective of this study is therefore to propose two alternative characterisations of the basic best–worst rule with a variable electorate and fixed alternatives.

First, we show that *reinforcement* and *top–bottom cancellation* imply *anonymity*. This result indicates that we can characterise the basic best–worst rule by *neutrality*, *continuity*, *reinforcement*, and *top–bottom cancellation*.

Second, we propose direct characterisation of the basic best–worst rule by *neutrality*, *reinforcement*, *top–bottom cancellation*, and, a new axiom, namely, *top–bottom non-negativity*, instead of

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[1] The *dis&approval* voting rule was analysed by Felsenthal (1989) and Alcantud and Laruelle (2014); it has a similar voting system. According to Alcantud and Laruelle (2014), application of the dis&approval voting rule assumes that we assign 1, $-1$, and 0 points for approval, disapproval, and other alternatives, respectively.

[2] See Garciá-Lapresta et al. (2010).

[3] See also Hansson and Sahlquist (1976).

https://doi.org/10.1016/j.econlet.2018.08.013
0165-1765/

*continuity*. Thus, we do not apply Young's (1975) theorem to the second characterisation. *Top–bottom non-negativity* requires that if the difference in number between the individuals choosing a certain alternative as their best and worst alternatives respectively is strictly less than 0, that alternative is not included in the social choice.

The remainder of this paper is structured as follows. Section 2 details our notations and definitions. Section 3 introduces the above axioms. Section 4 presents two characterisations of the basic best–worst rule.

## 2. Preliminaries

Let $N$ be a finite set of individuals such that $N \subset \mathbb{N}_+$ and $|N| \geq 1$, where $\mathbb{N}_+$ is the set of positive integers and $|N|$ indicates the cardinality of $N$. Now, suppose that $X$ is the finite set of all alternatives and $|X| \geq 2$. The alternatives in $X$ will be denoted by $a, b, c$, etc.

Next, suppose that $P_i \in \mathscr{D}_i \subseteq \mathscr{P}$ is a linear ordering over $X$ for each $i \in N$, where $\mathscr{P}$ is the set of all preference relations over $X$ and $\mathscr{D}_i$ is a set of feasible preference relations over $X$ for $i \in N$. Also, let $\mathcal{P} = (P_i)_{i \in N} \in \mathscr{D}$ be the profile of all $P_i$ such that $\mathscr{D} = \Pi_{i \in N} \mathscr{D}_i \subseteq \mathscr{P}^{|N|}$. For simplicity, we assume the following property in this study:

***Unrestricted domain*:** $\mathscr{D} = \mathscr{P}^{|N|}$.

Let $C : \mathcal{P} \rightrightarrows X$ be a *social choice correspondence*. Additionally, let $n_{aj}(\mathcal{P})$ be the number of individuals whose $j$th most preferred alternatives are $a$. Then, suppose that $\boldsymbol{n}_a(\mathcal{P}) = (n_{a1}(\mathcal{P}), \ldots, n_{a|X|}(\mathcal{P}))$.

We then introduce the *basic best–worst scoring vector*: $\boldsymbol{s}^{bw} = (1, 0, \ldots, 0, -1)$. Using $\boldsymbol{s}^{bw}$ and $\boldsymbol{n}_a(\mathcal{P})$, we can define the *basic best–worst score function* $f_a^{bw} : \mathscr{P}^{|N|} \to \mathbb{Z}$ for each alternative as follows: $f_a^{bw}(\mathcal{P}) = \boldsymbol{s}^{bw} \cdot \boldsymbol{n}_a(\mathcal{P})$,[4] where $\mathbb{Z}$ is the set of all integers. Finally, the *basic best–worst rule* is defined as given below.

**Definition 1.** Basic best–worst rule:

$C^{bw}(\mathcal{P}) = \{a \in X \mid a \in argmax_{b \in X} f_b^{bw}(\mathcal{P})\}$.

## 3. Axioms

The following six axioms characterise $C^{bw}$.

First, $C$ is *anonymous* if

$$C(\mathcal{P}) = C((P_{\pi(i)})_{i \in N})$$

for any $\pi \in \Pi$ and for any $\mathcal{P} \in \mathscr{P}^{|N|}$, where $\pi$ is a permutation such that $\pi : N \to N$, and $\Pi$ is the set of all permutations on the individuals in $N$. Thus, *anonymity* requires that no social choice should depend on the names of individuals.

Second, $C$ is *neutral* if

$$C(\psi(\mathcal{P})) = \{\lambda(a) \in X \mid a \in C(\mathcal{P})\}$$

for any $\lambda \in \Lambda$ and for any $\mathcal{P} \in \mathscr{P}^{|N|}$, where $\lambda$ is a permutation such that $\lambda : X \to X$, $\Lambda$ is the set of all permutations on the alternatives in $X$, $\psi$ is a permutation such that $\psi : \mathscr{P}^{|N|} \to \mathscr{P}^{|N|}$, and $\Psi$ is the set of all permutations on the preference profiles of the individuals in $N$. Note that for an arbitrary $N \subset \mathbb{N}_+$, every $\lambda \in \Lambda$ induces $\psi \in \Psi$.[5] Intuitively, *neutrality* requires that no social choice should depend on the names of alternatives.

Third, $C$ is *continuous* if

$$C((P_i)_{i \in mN_1 \cup N_2}) = C((P_i)_{i \in N_1})$$

for all $N_1, N_2 \subset \mathbb{N}_+$ such that $N_1 \cap N_2 = \emptyset$, where $m$ is a sufficiently large positive integer, and $mN$ indicates the union of the $m(\in \mathbb{N}_+)$ 'clone' sets of $N(\subset \mathbb{N}_+)$. Assume that every clone set has the same preference profile as $N$. Additionally, let each clone set be disjoint from $N$ and other clone sets. From the above statement, *continuity* requires that the social choice of a certain society $N \subset \mathbb{N}_+$ is equal to that of a unified society if the unified society consists of the following societies: (i) sufficiently many societies having the same preference profiles as $N$, and (ii) a society having a different preference profile.

Fourth, $C$ satisfies *reinforcement* if

$$C(\mathcal{P}_1) \cap C(\mathcal{P}_2) \neq \emptyset \Rightarrow C(\mathcal{P}_1) \cap C(\mathcal{P}_2) = C(\mathcal{P}_1 + \mathcal{P}_2)$$

for any $\mathcal{P}_1 = (P_i)_{i \in N_1} \in \mathscr{P}^{|N_1|}$ and for any $\mathcal{P}_2 = (P_i)_{i \in N_2} \in \mathscr{P}^{|N_2|}$ such that $N_1, N_2 \subset \mathbb{N}_+$ and $N_1$ and $N_2$ are disjoint, where $\mathcal{P}_1 + \mathcal{P}_2 = (P_i)_{i \in N_1 \cup N_2} \in \mathscr{P}^{|N_1|+|N_2|}$ for all $N_1, N_2 \subset \mathbb{N}_+$. Thus, this axiom requires that if the intersection of the social choices for any two disjoint subsets of individuals is not empty, then the intersection is equal to the social choice for the union of those two subsets.

Fifth, $C$ satisfies *top–bottom non-negativity* if

$$n_{a1}(\mathcal{P}) < n_{a|X|}(\mathcal{P}) \Rightarrow a \notin C(\mathcal{P})$$

for all $a \in X$ and for any $P \in \mathscr{P}$. This axiom requires that, for each alternative, if the number of individuals who have most preference for the alternative is less than the number of individuals who have least preference for the alternative, that alternative should not be included in the social choice. *Top–bottom non-negativity* implies *averseness*, as proposed in Kurihara (2018).[6]

Finally, $C$ satisfies *top–bottom cancellation* if

$$[n_{a1}(\mathcal{P}) = n_{a|X|}(\mathcal{P}) TIZ@D \forall a \in X] \Rightarrow C(\mathcal{P}) = X$$

for any $\mathcal{P} \in \mathscr{P}^{|N|}$. Hence, *top–bottom cancellation* requires that the social choice is $X$ if the number of individuals preferring $a$ as their best alternatives is cancelled out by the number of individuals preferring $a$ as their worst alternatives for every $a \in X$.

## 4. Results

Before characterising $C^{bw}$, Lemma 1 shows that *reinforcement* and *top–bottom cancellation* imply *anonymity*. The method to prove this is similar to that used in Young (1974): *reinforcement* and *cancellation*[7] imply *anonymity*.

**Lemma 1.** *If $C$ satisfies reinforcement and top–bottom cancellation, then $C$ is based only on the values of $n_{a1}(\mathcal{P}) - n_{a|X|}(\mathcal{P})$, $a \in X$.*

**Proof.** Suppose that $C$ satisfies *reinforcement* and *top–bottom cancellation*. Let $\mathcal{P}_1 = (P_i)_{i \in N_1}$ and $\mathcal{P}_2 = (P_i)_{i \in N_2}$, where $N_1, N_2 \subset \mathbb{N}_+$, such that

$$n_{a1}(\mathcal{P}_1) - n_{a|X|}(\mathcal{P}_1) = n_{a1}(\mathcal{P}_2) - n_{a|X|}(\mathcal{P}_2)$$

for all $a \in X$, and for all $a \in X$,

$$q_a = n_{a1}(\mathcal{P}_1) - n_{a1}(\mathcal{P}_2).$$

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4 The dot product is used in the formula.

5 There exists a bijection whose domain and co-domain are $\Lambda$ and $\Psi$, respectively. For example, assume that $X = \{a, b, c\}$, $N = \{1, 2\}$, and society has the following preference profile: $(P_1, P_2) \in \mathscr{P}^2$ such that $aP_1bP_1c$ and $bP_2cP_2a$. If $\lambda(a) = b$, $\lambda(b) = c$, and $\lambda(b) = c$, we obtain $\psi((P_1, P_2)) = (P_1', P_2') \in \mathscr{P}^2$ such that $bP_1'cP_1'a$ and $cP_2'aP_2'b$.

6 *Averseness* requires that if we assume that only one individual exists, then the individual's social choice will not include the worst alternative.

7 Young (1974) characterised the Borda rule by *neutrality*, *reinforcement*, *faithfulness*, and *cancellation*. *Faithfulness* requires that if there exists only one individual, then the best alternative for the individual will be the social choice. *Cancellation* requires that, for all $\mathcal{P} \in \mathscr{P}^{|N|}$, if $n_{ab}(\mathcal{P}) = n_{ba}(\mathcal{P})$ for all $a, b \in X$, then $C(\mathcal{P}) = X$, where $n_{ab}(\mathcal{P}) = |\{i \in N \mid aP_ib\}|$ for all $a, b \in X$.