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Finite powers of selectively pseudocompact groups ☆

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ABSTRACT

A space X is called *selectively pseudocompact* if for each sequence $(U_n)_{n<\omega}$ of pairwise disjoint nonempty open subsets of X there is a sequence $(x_n)_{n<\omega}$ of points in X such that $x_n \in U_n$, for each $n < \omega$, and $cl_X(\{x_n : n < \omega\}) \setminus (\bigcup_{n<\omega} U_n) \neq \emptyset$. Countably compact spaces are selectively pseudocompact and every selectively pseudocompact space is pseudocompact. We show, under the assumption of CH, that for every positive integer $k > 2$ there exists a topological group whose k-th power is countably compact but its $(k+1)$-st power is not selectively pseudocompact. This provides a positive answer to a question posed in [10] in any model of $ZFC + CH$.

1. Introduction

In this article, every space will be Tychonoff and every topological group will be Hausdorff (hence, they will be also Tychonoff). For an infinite set X, $[X]^{<\omega}$ will denote the family of all finite subsets of X and $[X]^{\omega}$ will denote the family of all countable infinite subsets of X. A finite set $\{x_0, ..., x_l\}$ of elements of an Abelian group G is called *independent* if does not contain 0 and if $\sum_{i\leq l} n_i x_i \neq 0$, then $n_i x_i = 0$ for each $i \leq l$. A nonempty subset X of G is called *independent* if every finite subset of X is independent. The continuum will be denoted by $\mathfrak{c}$. In this paper, we shall consider the group $[\mathfrak{c}]^{<\omega}$, with the symmetric

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difference as its group operation, as a vector space over the field $\{0,1\}$, and when we simply say that a subset of $[\mathfrak{c}]^{<\omega}$ is linearly independent, we shall understand that it is linearly independent in $[\mathfrak{c}]^{<\omega}$ as a vector space.

Two of the most outstanding generalizations of compactness have been countable compactness (introduced by M. Fréchet [6]) and pseudocompactness (introduced by E. Hewitt [16]). Since then many variations of these two properties have been considered in different contexts: in the book [17], the reader may find several topological properties, results and open problems related to pseudocompactness. In the realm of Tychonoff spaces, we know that a space X is countably compact iff every sequence of points has a cluster point. Hence, we observe that a space X is countably compact iff for every sequence $(A_n)_{n<\omega}$ of nonempty subsets of X there is a sequence $(x_n)_{n<\omega}$ of points in X such that $x_n \in A_n$, for each $n<\omega$, and $\{x_n : n<\omega\}$ has a cluster point in X. This makes natural to replace the term "subset" by the term "open subset" and thus we obtain the following generalization of countable compactness which was introduced in [8] with the name "strong pseudocompactness": this original name was changed in the paper [4] since the authors noticed that the term "strong pseudocompactness" was already used to name a different topological property.

Definition 1.1. A space X is called *selectively pseudocompact* if for each sequence $(U_n)_{n<\omega}$ of nonempty open subsets of X there is a sequence $(x_n)_{n<\omega}$ of points in X such that $x_n \in U_n$, for each $n<\omega$, and the set $\{x_n : n<\omega\}$ has a cluster point in X.

It is not hard to see that the definition of selective pseudocompactness given in Definition 1.1 is equivalent to the one stated in the abstract. It is evident that every countably compact space is selectively pseudocompact and every selectively pseudocompact space is pseudocompact. The authors of the paper [4] studied some variations of selective pseudocompactness and compared them with some standard notions.

There is a selectively pseudocompact group that is not countably compact and a pseudocompact group that is not selectively pseudocompact (for these two examples the reader is referred to [10]). However, the pseudocompact groups have a property very similar to selective pseudocompactness:

Theorem 1.2. *[8, Th. 2.4] For a topological group G, the following conditions are equivalent.*

(1) *The group G is pseudocompact.*
(2) *For each family $\{U_n : n<\omega\}$ of pairwise disjoint nonempty open subsets of G there is a discrete subset D of G contained in $\bigcup_{n<\omega} U_n$ such that $cl_G(D) \setminus \left(\bigcup_{n<\omega} U_n\right) \neq \emptyset$ and $|D \cap U_n| < \omega$, for every $n<\omega$.*
(3) *For each family $\{U_n : n<\omega\}$ of pairwise disjoint nonempty open subsets of G there is a discrete subset D of G contained in $\bigcup_{n<\omega} U_n$ such that $cl_G(D) \setminus \left(\bigcup_{n<\omega} U_n\right) \neq \emptyset$, $|D \cap U_n| \leq \omega$, for every $n<\omega$, and $D \cap U_n \neq \emptyset$ for infinitely many $n's$.*

A countably compact space whose square is not pseudocompact (for one of these spaces see [12]) is an example of a selectively pseudocompact space whose square is not pseudocompact. In the realm of topological groups, it is well-known that "the product of pseudocompact groups is pseudocompact", this fact was established by W.W. Comfort and K.A. Ross in their classical paper [1]. All these remarks suggest naturally the following question listed in [10] and in [9].

Question 1.3. Is selective pseudocompactness productive in the class of topological groups?

Let us make some comments about a possible solution to Question 1.3 inside of a model of ZFC. It is an old open problem posed by W.W. Comfort whether or not the product of two countably compact groups is countably compact. In 1980, E.K. van Douwen [5] assuming the existence of a countably compact Boolean group without non-trivial convergent sequences constructed, in ZFC, two countably compact