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Unified formulation of the momentum-weighted interpolation for collocated variable arrangements

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ABSTRACT

Momentum-weighted interpolation (MWI) is a widely used discretisation method to prevent pressure–velocity decoupling in simulations of incompressible and low Mach number flows on meshes with a collocated variable arrangement. Despite its popularity, a unified and consistent formulation of the MWI is not available at present. In this work, a discretisation procedure is devised following an in-depth analysis of the individual terms of the MWI, derived from physically consistent arguments, based on which a unified formulation of the MWI for flows on structured and unstructured meshes is proposed, including extensions for discontinuous source terms in the momentum equations as well as discontinuous changes of density. As shown by the presented analysis and numerical results, the MWI enforces a low-pass filter on the pressure field, which suppresses oscillatory solutions. Furthermore, the numerical dissipation of kinetic energy introduced by the MWI is shown to converge with third order in space and is independent of the time-step, if the MWI is derived consistently from the momentum equations. In the presence of source terms, the low-pass filter on the pressure field can be shaped by a careful choice of the interpolation coefficients to ensure the filter only acts on the driving pressure gradient that is associated with the fluid motion, which is shown to be vitally important for the accuracy of the numerical solution. To this end, a force-balanced discretisation of the source terms is proposed, that precisely matches the discretisation of the pressure gradients and preserves the force applied to the flow. Using representative test cases of incompressible and low Mach number flows, including flows with discontinuous source terms and two-phase flows with large density ratios, the newly proposed formulation of the MWI is favourably compared against existing formulations and is shown to significantly reduce, or even eliminate, solution errors, with an increased stability for flows with large density ratios.

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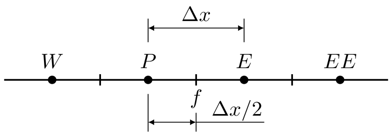

Fig. 1. One-dimensional example of an equidistant mesh, where Δx is the mesh spacing.

1. Introduction

The coupling of pressure and velocity is a key difficulty of simulating incompressible flows and has been a central topic of computational fluid dynamics (CFD) for the past decades [1–3]. The difficulties associate with the pressure–velocity coupling can be illustrated by assuming an isothermal, incompressible flow, which is governed by the momentum equations

$$\rho\left(\frac{\partial u_j}{\partial t}+\frac{\partial u_j u_i}{\partial x_i}\right)=-\frac{\partial p}{\partial x_j}+\frac{\partial \tau_{ij}}{\partial x_i}+S_j \tag{1}$$

and the continuity equation

$$\frac{\partial u_i}{\partial x_i}=0\,, \tag{2}$$

where ρ is the density, $\boldsymbol{u}$ the velocity, p is the pressure, $\boldsymbol{\tau}$ is the shear stress tensor, $\boldsymbol{S}$ are the source terms, t is time and $\boldsymbol{x}$ is the coordinate axis. Aside from the question of how to solve the strongly coupled pressure and velocity fields, the governing equations of a three-dimensional incompressible flow only provide three independent equations for four unknowns (three velocity components plus pressure), which makes the formulation of an equation for pressure based on the governing flow equations non-trivial and has lead to a variety of segregated [1,3,4] and coupled [5–7] algorithms. Furthermore, discretising the pressure gradient on the one-dimensional equidistant mesh shown in Fig. 1 using central differencing yields

$$\left.\frac{\partial p}{\partial x}\right|_P \approx \frac{p_E-p_W}{2\Delta x}\,, \tag{3}$$

where Δx is the mesh spacing. The pressure gradient at node P is, crucially, not dependent on the pressure value at node P, irrespective of the algorithm applied to solve the governing equations. Consequently, the governing equations permit two independent pressure fields in a *chequerboard* pattern [3,4] as a valid solution to the discrete equations, a result that naturally extends to higher dimensions.

Pressure–velocity decoupling is a discretisation issue typically associated with incompressible flows. When compressible flows are considered, most numerical frameworks use density as a primary variable, while pressure is determined indirectly via an appropriate equation of state. Although such density-based algorithms are the method of choice when the compressibility of the flow is appreciable, they are ill-suited for flows with low Mach numbers [2,8], in particular in the incompressible limit. Motivated by the desire to compute flows at all speeds with the same numerical framework, a number of pressure-based algorithms for flows at all speeds have been developed, *e.g.* [9–12]. However, the insignificant compressibility of flows with low Mach number admits pressure–velocity decoupling in the compressible flow solution on meshes with collocated variable arrangement.

Historically, pressure and velocity were coupled by staggering the points at which pressure and velocity are evaluated, the *staggered variable arrangement*, as proposed by Harlow and Welch [13], with velocity typically evaluated at the centres of the cell-faces, while all other variables are evaluated and stored at the cell centres. A staggered variable arrangement enforces a natural coupling between pressure and velocity, and yields a very compact stencil for the pressure gradient that drives the velocity at the adjacent cell centres through the momentum equations. There is no doubt that for Cartesian meshes, a staggered variable arrangement is efficient and effective. However, as CFD has matured as a tool, it has found ever more frequent application to analyse flows in complex geometries, represented by unstructured meshes, for which the application of a staggered variable arrangement is difficult and may include complex corrections to account for meshes of relatively poor quality [14,15]. This difficulty, in conjunction with the bookkeeping overhead associated with staggered variable arrangements [16], has motivated the development of discretisation methods for *collocated variable arrangements*, in which all variables are stored at cell centres, that prevent the pressure–velocity decoupling ensuing as a result of the scenario presented in Eq. (3). Notable methods that allow robust computations on meshes with collocated variable arrangement are the *momentum-weighted interpolation* (MWI), based on the work of several researchers in the early 1980s [17] and widely attributed to Rhie and Chow [18], the artificial compressibility method [19] and one-sided differencing [20], of which MWI is by far the most widely used at present [21].

The principle of the MWI, also frequently referred to as *pressure-weighted interpolation* or *Rhie–Chow interpolation*, is to evaluate the velocity at the faces based on weighting coefficients that are derived from the discretised momentum equations, including pressure gradients. By construction, the MWI emulates a staggered variable arrangement, introducing a cell-to-cell