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Time and magnitude monitoring based on the renewal reward process

Sajid Ali [a,b,*], Antonio Pievatolo [c]

[a] Department of Statistics, Quaid-i-Azam University, Islamabad 45320, Pakistan
[b] Department of Decision Sciences, Bocconi University, via Röentgen 1, Milan 20136, Italy
[c] CNR-IMATI, via Bassini 15, Milan 20133, Italy

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ABSTRACT

Time between event (TBE) charts are SPC tools for monitoring the occurrence of unwanted events, such as the appearance of a defective item or a failure of a piece of equipment. In some cases, a magnitude, indicating the severity of the event, is also measured. Time and magnitude charts, which are based on the assumption that the stochastic process underlying the occurrence of events is the marked Poisson process, are the preferred option. However, these charts are not suitable to deal with damage events caused by repeatedly occurring shocks or stress conditions. To bridge this gap, we introduce a new control chart based on the assumption of a renewal process with rewards, where the reward represents magnitude, and a magnitude-over-threshold condition represents the occurrence of an event. In particular, we consider two cases for magnitude: (i) magnitude is cumulative over time and (ii) magnitude is non-cumulative or independent over time. We use known results in renewal theory to provide expressions of the probability distributions needed to compute the control limits and perform a simulation analysis of the control chart performance.

1. Introduction

Statistical quality control is a collection of statistical methods, which are used to monitor and improve the quality of a process. Currently, statistical quality control is not limited to the manufacturing industry, but is also used in environmental science, biology, genetics, epidemiology, medicine, finance, law enforcement and athletics. Among the SPC tools, control charts are probably the most technically sophisticated. One of the main purposes of control charts is to distinguish between the variation due to chance causes and the variation due to assignable causes in order to prevent overreaction and under-reaction to the process (cf. [1]).

There are different types of control charts in the literature to handle different situations [2]. A special type of control chart called time-between-events (TBE) is used to monitor rare events or the so-called high-quality processes. Traditional TBE charts considered the time interval X between the occurrences of an event by completely ignoring the magnitude M associated with it, representing the size of the event itself. However, there are many real applications where both time and magnitude are important and ignoring one of them leads to misleading conclusions. If the magnitude is also available, it has been recognised that a joint monitoring of time and magnitude improves the performance of the control chart. For example, Wu et al. [3] proposed control charts for the combined monitoring of TBE and magnitude, providing a decision rule based on both individual X and M charts. Later, Wu et al. [4] introduced the rate chart to monitor magnitude and TBE data by considering their ratio, finding that it is more effective than the individual X or M charts and also than the combined monitoring chart. The rate chart with an integer magnitude was proposed by Liu et al. [5]. Liu et al. [6] proposed a joint control chart by considering a truncated Poisson distribution for the magnitude, finding that the new chart outperforms the individual charts. Recently, Qu et al. [7] also introduced a time and magnitude chart by assuming an exponential distribution for time and a normal distribution for magnitude. The authors compared the proposed chart with the existing ones, i.e., time, magnitude, time and magnitude, and rate charts, and showed that the new chart is more efficient. Some works related to CUSUM charts are Wu et al. [8], Qu et al. [9], and Qu et al. [10]. We refer to Ali et al. [11] for a detailed review about time and magnitude control charts.

The importance of control charts for reliability data has been highlighted by Xie et al. [12]. More recently, Vining et al. [13] observed that there is a need to develop process control techniques for reliability data to ensure that a product or a process maintains the expected reliability standard. In this paper we are concerned with damage events caused by randomly occurring shocks or stress conditions, which eventually lead to a failure event or require repair when the magnitude of the damage has

* Corresponding author at: Department of Statistics, Quaid-i-Azam University, Islamabad 45320, Pakistan.
E-mail addresses: sajidali@qau.edu.pk (S. Ali), antonio.pievatolo@mi.imati.cnr.it (A. Pievatolo).

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crossed an appropriate threshold. This event can be categorized into two failure modes: catastrophic failure, in which the failure occurs by some sudden shock, and cumulative shock failure, in which the failure occurs by physical deterioration due to age or cumulative wear. The two cases will be called *independent damage process* and *cumulative damage process*, respectively. The renewal process with rewards is suitable to describe both, by considering as the time for the occurrence of an event the first passage time above the threshold either by a single reward or by the accumulated rewards. When a magnitude-over-threshold event happens, the product or process are renewed and the next TBE is the next first passage time, so that this chart can be called an FPT-chart. By monitoring this TBE, a reliability engineer can assess whether a product is being used according to its design limits or a process is being run according to specifications. A too frequent occurrence of magnitude-over-threshold events could also indicate deficiencies in the material used for the product or for the equipment involved in the process. A simple example of an independent damage process and of cumulative damage process is provided by Gut and Hüsler [14]: a material, such as a rope or a wire, can break due to fatigue because of the cumulative effect of loads within design limits after a long period of time or because a sudden big load exceeding its capacity; capacity could as well be lower than expected due to faulty material.

The existing time and magnitude control charts are based on the marked Poisson process, although this is not always explicitly stated. The renewal reward assumption is a generalisation in the direction of any lifetime distribution for the occurrence of shocks. However, time and magnitude charts monitor every single magnitude and TBE, not first passage times, therefore the FPT-chart cannot be viewed as a direct generalisation, rather as a complement to these. For example, if the magnitude is not directly observable and a shock process is in effect (be it independent or cumulative), then the failure time is a first passage time and it has a non-exponential distribution even in the simplest settings. In this case time and magnitude charts cannot be applied, while the FPT-chart is still usable because it is not necessary to establish a threshold to observe failures. If the magnitude is observable, both charts can be applied, but they are expected to react differently to the same changes in the underlying process. In case of a zero threshold the FPT-chart is a simple TBE chart.

This study is organized as follows. In Section 2 we define the renewal reward process formally and provide expressions for the first passage time distributions. The compound Poisson process is also mentioned as a special case of the renewal reward process. The FPT-chart construction and a numerical study of performance measures are given in Section 3. Also, a comparison of FPT charts with rate charts is presented in Section 3. An implementation of the FPT-chart is the subject of Section 4. Section 5 contains a brief summary of the outcomes, conclusions and suggestion for future studies.

2. Cumulative and independent processes

In this section, we shall introduce the necessary definitions and formulas that are required for the development of FPT-charts, obtained from Nakagawa [15] and Nakagawa [16].

We denote by $N(t)$ a counting process, by X_i the TBE and by M_i the magnitude associated with X_i for $i \geq 1$.

Definition 2.1. A counting process $\{N(t),\ t \geq 0,\ t \in T\}$ with independent and identically distributed (iid) inter-arrival times $X_1, X_2, \dots$ with a common distribution F is called a renewal process.

Definition 2.2. Let $N(t)$ be a renewal process and let M_i denote the reward (such as damage, wear, fatigue, or cost) that is attached to each inter-arrival time X_i. If the pairs (X_i, M_i) for $i = 1, 2, \dots$ are independent and identically distributed, then the stochastic process $Y(t) = \sum_{i=1}^{N(t)} M_i$ is called a renewal reward process.

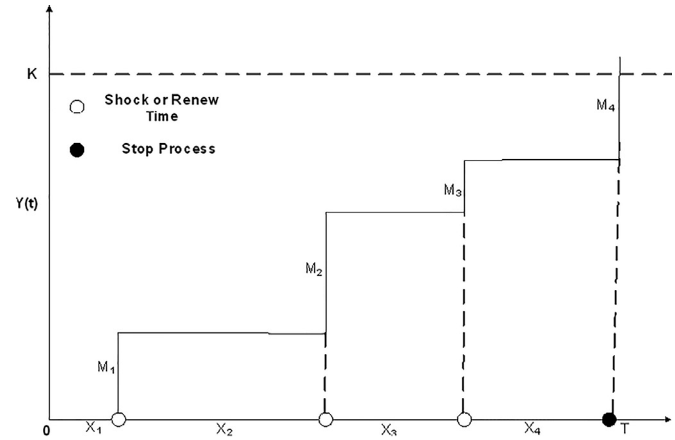

Fig. 1. Process for a standard cumulative damage model.

Therefore, the renewal reward assumption is a generalization of the marked Poisson process.

Let $F(x) = Pr\{X_i \leq x\}$ and $G(m) = Pr\{M_i \leq m\}$ be the cumulative distribution functions of X_i and M_i, respectively, with finite means. In addition, suppose K is a fixed threshold for the damage. In the cumulative damage scenario, the FPT-chart is based on the distribution of Z, the first passage time: $Pr\{Z \leq t\}$ where $Z = \min_t\{Y(t) > K\}$. For the independent damage scenario, the first passage time can be defined as $Z = \sum_{j=1}^{i^*} X_j$, where $i^* = \min\{j = 1, 2, 3, \dots \mid M_j > K\}$. We remark that K can be only implied if the damage due to shocks is not observable and Z represents the time of an observable failure.

In the following subsections we provide expressions for the distribution of Z, including also the homogeneous compound Poisson process, as a special case of the renewal reward process.

2.1. Cumulative damage process

Definition 2.1.1. Let $(X_i,\ M_i)$ denote a sequence of times between shocks (X_i) with an associated damage (M_i) undergone by a unit or system. Suppose that each damage is additive and the system or unit fails when the total damage has exceeded a failure threshold K where $0 < K < \infty$, for the first time (cf. Fig. 1). A process with such a behavior is called a cumulative damage process.

If $Y(t)$ is a renewal reward process, then the distribution of the first passage time Z is

$$\phi(t) = Pr\{Z \leq t\} = \sum_{n=0}^{\infty}[G^{(n)}(K) - G^{(n+1)}(K)]F^{(n+1)}(t)\ . \tag{1}$$

2.2. Independent damage process

Definition 2.2.1. Let $(X_i,\ M_i)$ denote a sequence of times between shocks (X_i) with an associated damage (M_i) undergone by a unit or system. Suppose that the damage is not additive and the system or unit fails the first time the amount of damage a threshold level K. This type of process is called an independent damage model (cf. Fig. 2).

The first passage time distribution is

$$Pr\{Z \leq t\} = \sum_{n=0}^{\infty}[G^n(K) - G^{n+1}(K)]F^{(n+1)}(t)\ . \tag{2}$$

Notice that Eq. (2) does not have the convolution (.) for the magnitude distribution as compared to Eq. (1).