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Metric reduction and generalized holomorphic structures

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ABSTRACT

In this paper, metric reduction in generalized geometry is investigated. We show how the Bismut connections on the quotient manifold are obtained from those on the original manifold. The result facilitates the analysis of generalized Kähler reduction, which motivates the concept of metric generalized principal bundles and our approach to construct a family of generalized holomorphic line bundles over $\mathbb{C}P^2$ equipped with some non-trivial generalized Kähler structures.

1. Introduction

Generalized complex geometry initiated by N. Hitchin and his school is a simultaneous generalization of symplectic geometry and complex geometry. Since Marsden–Weinstein reduction is a basic construction in symplectic geometry, it is natural to explore a generalized version of symplectic reduction in generalized geometry. This topic was treated in great generality in the formalism of Courant reduction in [1]. When furthermore there is a generalized metric on the Courant algebroid to be reduced, it also descends to the reduced Courant algebroid under proper conditions. In [2], this 'metric reduction' was investigated; in particular, this procedure was checked from the angle of geometry of tangent bundles. The present paper arises from our work [3] on trying to understand metric reduction from a topological field theoretic viewpoint.

Considerations in generalized geometry are conceptually direct and useful, but the underlying structures often hide in depth and need careful analysis. For example, generalized Kähler reduction is easily understood from the general procedure of reduction of Dirac structures, but it contains some sophisticated details from the viewpoint of classical complex geometry. Some of these were included in [2]. In this paper, we will carry on this investigation.

We pay much attention on the special case of *isotropic trivially extended G-actions* in the sense of [1], where G is a compact connected Lie group. With an invariant generalized metric in place, the manifold M under consideration carries two horizontal distributions $\tau_\pm$, which are central in our paper. Basically, they are used to express the Bismut connections in the reduced manifold $M_{red} := M/G$ in terms of Bismut connections in M. This is different from the case of reducing the Levi-Civita connection on M–In the latter case, a connection of the principal bundle $M \to M_{red}$ naturally arises from the G-invariant metric g, i.e. the horizontal distribution is just the orthogonal complement $\mathcal{H}$ of the vertical distribution. The Levi-Civita connection on M_{red} can then be expressed using the Levi-Civita connection on M and the orthogonal projection

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from TM to $\mathcal{H}$. As for reducing Bismut connections, it is not as directly solved as in the ordinary case and should be motivated by conceptual considerations in generalized geometry. This investigation of reducing Bismut connections is motivated by gauging a zero-dimensional supersymmetric σ-model in [3].

When the invariant generalized metric is from a generalized Kähler manifold $\mathcal{M}$, the situation becomes more interesting. To get a reduced generalized Kähler manifold, an invariant submanifold $M \subset \mathcal{M}$ should be carefully chosen and the reduced generalized Kähler structure will then sit on $M_{red} = M/G$. Hence M only serves as an intermediate object in this procedure. But in this paper M as a *metric generalized principal bundle* (see Section 5) proves to have its own interest: The curvatures of $\tau_\pm$ are of type $(1, 1)$ w.r.t. the reduced complex structures $\tilde{J}_\pm$ on M_{red} respectively. Thus any associated complex vector bundle acquires simultaneously a $\tilde{J}_+$-holomorphic structure and a $\tilde{J}_-$-holomorphic structure.[1] This motivates our approach to constructing generalized holomorphic vector bundles from generalized Kähler reduction.

The paper is organized as follows. In Section 2, we review the basic content of generalized geometry. The goal of Section 3 is to lay the concrete background for later development by investigating the notion of isotropic trivially extended G-action in the presence of an invariant generalized metric. Compared with the work in [2], we hardly contain much essentially new content, but our viewpoint is slightly different. In particular, we include some details of the reduced structures which were missing in [2], and emphasize the basic role of the distributions $k_\pm$ (Eq. (3.2) is essential for reducing the Bismut connections) which was not explicitly mentioned in [2]. In Section 4, we mainly tackle the problem of expressing the reduced Bismut connections in terms of Bismut connections in the original manifold (Theorem 4.1). The curvature of the reduced Bismut connection is also computed in terms of the reduction data (Theorem 4.3). These computations play a basic role in [3]. The last three sections devote to using generalized Kähler reduction to produce generalized holomorphic vector bundles. Section 5 discusses the notion of metric generalized principal G-bundle and its associated relative curvature. Section 6 revisits generalized Kähler reduction in the spirit of previous sections, and emphasis is put on structures on the intermediate metric generalized principal G-bundle, which carries a biholomorphic structure. These two sections pave the way for us to produce generalized holomorphic vector bundles via generalized Kähler reduction in Section 7. We give a sufficient condition for the biholomorphic structure to be generalized holomorphic in the Hamiltonian case. As examples, we have constructed generalized holomorphic line bundles on $\mathbb{C}P^2$ equipped with non-trivial generalized Kähler structures.

2. Basics of generalized geometry

In this section, we collect the most relevant aspects of generalized geometry. For a detailed account for it, we refer the reader to [4,5].

In generalized geometry, one considers geometric structures defined on the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$ of a smooth manifold M, or more generally on an exact Courant algebroid over M.

A Courant algebroid E is a real vector bundle E over M, together with an anchor map π to TM, a non-degenerate inner product and a so-called Courant bracket $[\cdot, \cdot]_c$ on $\Gamma(E)$. These structures should satisfy some compatibility axioms. E is called exact, if the short sequence

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0$$

is exact. In this paper, by 'Courant algebroid', we always mean an exact one. Given E, one can always find an isotropic right splitting $s : TM \to E$, which has a curvature form $H \in \Omega^3_{cl}(M)$ defined by

$$H(X, Y, Z) = \langle [s(X), s(Y)]_c, s(Z)\rangle, \quad X, Y, Z \in \Gamma(TM).$$

By the bundle isomorphism $s + \pi^* : TM \oplus T^*M \to E$, the Courant algebroid structure can be transported onto $\mathbb{T}M$. Then the inner product $\langle \cdot, \cdot \rangle$ is the natural pairing, i.e. $\langle X + \xi, Y + \eta\rangle = \xi(Y) + \eta(X)$, and the Courant bracket is

$$[X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X\eta - \iota_Y d\xi + \iota_Y\iota_X H, \tag{2.1}$$

called the H-twisted Courant bracket. Different splittings are related by B-field transforms, i.e. $e^B(X + \xi) = X + \xi + \iota_X B$, where B is a 2-form.

A maximal isotropic subbundle $L \subset E$ is called an almost Dirac structure. If L is involutive w.r.t. the Courant bracket, it is called a Dirac structure. These notions can be extended directly to the complexified setting which interests us most.

Definition 2.1. A generalized complex structure on E is a complex structure $\mathbb{J}$ on E orthogonal w.r.t. the inner product and whose $\sqrt{-1}$-eigenbundle $L \subset E \otimes \mathbb{C}$ is a complex Dirac structure.

Since $\mathbb{J}$ and its $\sqrt{-1}$-eigenbundle L are equivalent notions, we shall use them interchangeably to denote a generalized complex structure. At a point $x \in M$, the codimension of $\pi(L_x)$ in $T_xM \otimes \mathbb{C}$ is called the type of $\mathbb{J}$ at x. Type can vary along some subset of M, which makes the local geometry of generalized complex structures rather non-trivial.

A generalized complex structure L is an example of complex Lie algebroids. Via the inner product, $\wedge^\cdot L^*$ can be identified with $\wedge^\cdot \bar{L}$, and we have an elliptic differential complex $(\Gamma(\wedge^\cdot \bar{L}), d_L)$, which induces the Lie algebroid cohomology associated with the Lie algebroid L. The differential complex can be twisted by an L-module.

[1] Similar phenomenon, of course, occurs in ordinary Kähler reduction but is seldom emphasized in the literature.