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A representation theorem for measurable relation algebras ☆

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ABSTRACT

A relation algebra is called measurable when its identity is the sum of measurable atoms, where an atom is called measurable if its square is the sum of functional elements.

In this paper we show that atomic measurable relation algebras have rather strong structural properties: they are constructed from systems of groups, coordinated systems of isomorphisms between quotients of the groups, and systems of cosets that are used to "shift" the operation of relative multiplication. An atomic and complete measurable relation algebra is completely representable if and only if there is a stronger coordination between these isomorphisms induced by a scaffold (the shifting cosets are not needed in this case). We also prove that a measurable relation algebra in which the associated groups are all finite is atomic.

1. Introduction

The well-known pair of papers [8] and [9], by Jónsson and Tarski, were motivated by Tarski's efforts to prove that every model of his axiomatization of the calculus of relation algebras is representable, that is to say, every (abstract) relation algebra is isomorphic to a set relation algebra consisting of a universe of (binary) relations on some base set, under the standard set-theoretic operations on such relations. In the second of these papers, the authors proved several representation theorems for limited classes of relation algebras. In particular, they proved that an atomic relation algebra in which the atoms satisfy a specific "singleton inequality" is isomorphic to a set relation algebra. The singleton inequality is an inequality that is satisfied by a non-empty relation R and its converse in a set relation algebra of all binary relations on

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a base set if and only if R is a singleton relation in the sense that it has the form $R = \{(p, q)\}$ for some elements p and q in the base set.

Maddux [10] considerably strengthened this representation theorem. He eliminated the assumptions that the given relation algebra be atomic and that every atom satisfy the singleton inequality. Instead, he assumed only that the identity element be the sum of a set of non-zero *elements* satisfying the singleton inequality. Actually, he proved an even stronger version of this theorem by showing that every relation algebra in which the identity element is the sum of a set of non-zero elements satisfying the singleton inequality or a corresponding "doubleton inequality" is isomorphic to a set relation algebra. He called such relation algebras *pair dense*.

The results in this paper have been inspired by Maddux's pair-dense relation algebras. At first sight, it seems that the number 2 is the "end" of this kind of investigations, because there is no "tripleton inequality" that would characterize relations consisting of at most 3 pairs. This obstacle may be overcome by allowing oneself to use formulas from first-order logic instead of just equations and inequalities. In [5], an atom $x \leq 1'$ is defined to be *measurable* if the square $x ; 1 ; x$ is the sum of a set of *functions*, that is to say, a set of abstract elements f satisfying the functional inequality $f^{\smile} ; f \leq 1'$. These functions turn out to be abstract versions of permutations, and the set of these permutations that are non-zero and below the square $x ; 1 ; x$ form a group. The size of the group gives a measure of the size of x. A relation algebra is said to be *measurable* if the identity element is the sum of measurable atoms, and *finitely measurable* if all of the atoms in this sum have finite measure.

In [4], a large class of examples of measurable set relation algebras is constructed using systems of groups and corresponding systems of isomorphisms between quotients of the groups. The resulting algebras are called (*generalized*) *group relation algebras*, and every such algebra is an example of a complete and atomic measurable relation algebra. The class of these examples, however, does not comprehend all possible examples of complete and atomic measurable relation algebras. In [1], the class of examples is expanded by using systems of cosets to "shift", or change the value, of the operation of relational composition in group relation algebras. A characterization is given in [1] of when such "shifted" group relation algebras are relation algebras, and therefore examples of complete and atomic measurable relation algebras. They are called *coset relation algebras*. An example is given in [1] of a coset relation algebra—and therefore of an atomic, measurable relation algebra—that is not isomorphic to any set relation algebra, so not all atomic measurable relation algebras are representable in the classical sense of the word.

The purpose of the present paper is to prove that the class of coset relation algebras is adequate for the task of "representing in a wider sense" all atomic, measurable relation algebras. In the main theorem of the paper (Theorem 7.4), we show that every atomic, measurable relation algebra $\mathfrak{B}$ is *essentially isomorphic* to a coset relation algebra $\mathfrak{C}$ in the sense that the completion (the minimal complete extension) of $\mathfrak{B}$ is isomorphic to $\mathfrak{C}$. (The passage to the completion does not change the structure of $\mathfrak{B}$, it only fills in any missing infinite sums that are needed in order to obtain isomorphism with the complete relation algebra $\mathfrak{C}$.) In particular, every measurable relation algebra that is finite is isomorphic to a coset relation algebra. If the algebra $\mathfrak{B}$ is not finite, but is finitely measurable, then the assumption that $\mathfrak{B}$ be atomic may be dropped (Theorem 8.3). We also prove that a measurable relation algebra $\mathfrak{B}$ is essentially isomorphic to a group relation algebra if and only if $\mathfrak{B}$ has a "scaffold" of atoms, and this occurs if and only if $\mathfrak{B}$ is completely representable in the classical sense (Theorem 7.6–Corollary 7.9).

The structure of the paper is as follows. The definition of a relation algebra, and the relatively few basic relation algebraic laws that are needed to follow the proofs in the paper are presented in Section 2. Measurable atoms are investigated in Sections 3–5 in arbitrary, not necessarily measurable relation algebras. In Section 3, we define and investigate the group G_x associated to a measurable atom x. In Section 4, in the focus of investigation are elements with domain and range measurable atoms, in particular we investigate a special class of elements called regular elements. In Section 5, we show that regular elements with normal stabilizers induce isomorphisms between the corresponding quotient groups. In Section 6, we begin to work

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