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Gap-planar graphs ☆

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ABSTRACT

We introduce the family of *k-gap-planar graphs* for $k \geq 0$, i.e., graphs that have a drawing in which each crossing is assigned to one of the two involved edges and each edge is assigned at most k of its crossings. This definition is motivated by applications in edge casing, as a k-gap-planar graph can be drawn crossing-free after introducing at most k local gaps per edge. We present results on the maximum density of k-gap-planar graphs, their relationship to other classes of beyond-planar graphs, characterization of k-gap-planar complete graphs, and the computational complexity of recognizing k-gap-planar graphs.

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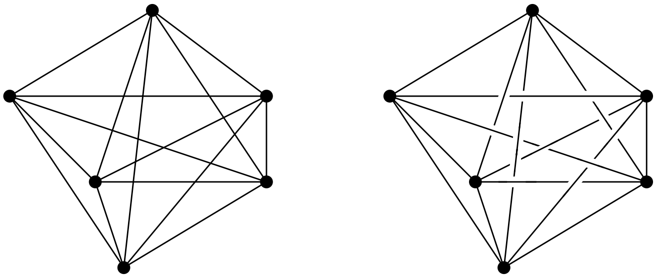

Fig. 1. A drawing of a graph G (left) and its cased version where each edge is interrupted at most twice, i.e., a 2-gap-planar drawing of G (right).

1. Introduction

Minimizing the overall number of edge crossings in a drawing has been the main objective of a large body of literature concerning the design of algorithms to automatically draw a graph. In fact, several graph drawing algorithms assume the input graph to be planar or planarized (that is, crossings are replaced with dummy vertices which are removed in a post-processing step). More recently, cognitive experiments suggested that the absence of specific kinds of edge crossing configurations has a positive impact on the human understanding of a graph drawing [36]. These practical findings motivated a line of research, commonly called *beyond planarity*, whose focus is on non-planar graphs that can be drawn by locally avoiding specific edge crossing configurations or by guaranteeing specific properties for the edge crossings (see, e.g., [11,33,35,41]).

Among the most investigated families of beyond-planar graphs are: *k-planar graphs* (see, e.g., [12,39,43]), which can be drawn with at most k crossings per edge; *k-quasiplanar graphs* (see, e.g., [2,3,25]), which can be drawn with no k pairwise crossing edges; *fan-planar graphs* (see, e.g., [9,13,37]), which can be drawn such that each edge is crossed by a (possibly empty) set of edges that have a common endpoint on one side; *RAC graphs* (refer, e.g., to [20]), which admit a straight-line drawing with right-angle crossings.

In this paper we introduce a family that generalizes k-planar graphs by introducing a nonsymmetric constraint on the intersection pattern of the edges. Intuitively speaking, we charge each crossing to only one of the two edges involved in the crossing and do not allow an edge to be charged many times. This constraint is motivated by *edge casing*, a method commonly used to alleviate the visual clutter generated by crossing lines in a diagram [5,24]. In a *cased drawing* of a graph, each crossing is resolved by locally interrupting one of the two crossing edges; see Fig. 1 for an illustration. This edge casing makes only *one* of the edges involved in the crossing hard to follow whereas the other one is unaffected. Regardless of the number of crossings, the drawing will remain clear as long as no edge is cased many times; thus, an edge could participate in arbitrarily many crossings as long as the other edges are cased. Eppstein et al. [24] studied several optimization problems related to edge casing, assuming the input is a graph together with a fixed drawing. In particular, the problem of minimizing the maximum number of gaps per edge in a drawing can be solved in polynomial time (see also Section 2). We also note that a similar drawing paradigm is used by *partial edge drawings (PEDs)*, in which the central part of each edge is erased, while the two remaining stubs are required to be crossing-free (see, e.g., [16,17]).

We formalize this idea with the family of *k-gap-planar graphs*, a family of graphs that can be drawn in the plane such that each crossing is assigned to one of the two involved edges and each edge is assigned at most k crossings (for some constant k). We present a rich set of results for k-gap-planar graphs related to classic research questions, such as bounds on the maximum density, drawability of complete graphs, complexity of the recognition problem, and relationships with other families of beyond-planar graphs. Our results can be summarized as follows:

- Every k-gap-planar graph with n vertices has $O(\sqrt{k} \cdot n)$ edges (Section 3). If $k = 1$, we prove an upper bound of $5n - 10$ for the number of edges in a 1-gap-planar multigraph with n vertices (without homotopic parallel edges), and construct 1-gap-planar (simple) graphs that attain this bound for all $n \geq 20$. Note that the same density bound is known to be tight for 2-planar graphs [43].
- We study relationships between the class of k-gap-planar graphs and other classes of beyond-planar graphs. For all $k \geq 1$, the class of $2k$-planar graphs is properly contained in the class of k-gap-planar graphs, which in turn is properly contained in the $(2k + 2)$-quasiplanar graphs (Section 4). We note that k-planar graphs are known to be $(k + 1)$-quasiplanar [4,31]. Furthermore, we investigate the relationship between k-gap-planar graphs and d-degenerate crossing graphs, a class of graphs recently introduced by Eppstein and Gupta [23].
- The complete graph K_n is 1-gap-planar if and only if $n \leq 8$ (Section 5).
- Deciding whether a graph is 1-gap-planar is NP-complete, even when the drawing of a given graph is restricted to a fixed rotation system that is part of the input (Section 6). Note that analogous recognition problems for other families of beyond-planar graphs are also NP-hard (see, e.g., [7,9,13,14,29,40]), while polynomial algorithms are known in some restricted settings (see, e.g., [6,9,14,19,22,34,32]).

Preliminaries and basic results are in Section 2. Conclusions and open problems are discussed in Section 7.