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Passive tracer in non-Markovian, Gaussian velocity field

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ABSTRACT

We consider the trajectory of a tracer that is the solution of an ordinary differential equation $\dot{X}(t) = V(t, X(t))$, $X(0) = 0$, with the right hand side, that is a stationary, zero-mean, Gaussian vector field with incompressible realizations. It is known, see Fannjiang and Komorowski (1999), Carmona and Xu (1996) and Komorowski et al. (2012), that $X(t)/\sqrt{t}$ converges in law, as $t \to +\infty$, to a normal, zero mean vector, provided that the field $V(t, x)$ is Markovian and has the spectral gap property. We wish to extend this result to the case when the field is not Markovian and its covariance matrix is given by a completely monotone Bernstein function.

1. Introduction and some assumptions

In this paper we would like to show the central limit theorem for a passive tracer model, when the velocity field is non-Markovian but Gaussian and exponentially mixing in time.

Passive tracer model is given by the following equation,

$$\begin{cases} \dfrac{dX(t)}{dt} = V\left(t, X(t)\right), \quad t > 0, \\ X(0) = 0, \end{cases} \tag{1.1}$$

where $V : \mathbb{R}^{1+d} \times \Omega \to \mathbb{R}^d$ is a real, $d-$dimensional, incompressible i.e. $\sum_{p=1}^{d} \partial_{x_p} V_p(t, x) \equiv 0$, zero mean, Gaussian random vector field over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

This model describes a trajectory of particle motion in an incompressible, disordered flow and has applications e.g. in turbulent diffusion and stochastic homogenization, see Majda and Kramer (1999), Kraichnan (1970), Warhaft (0000) and Sreenivasan and Schumacher (2010). Some basic problems concerning the asymptotic behavior of the tracer are: the law of large numbers (LLN) i.e. whether $X(t)/t$ converges to a constant vector v_* (called the *Stokes drift*), as $t \to +\infty$ and the central limit theorem (CLT), i.e. whether $(X(t) - v_* t)/\sqrt{t}$ is convergent in law to a normal vector $N(0, \kappa)$. The covariance matrix $\kappa = [\kappa_{ij}]_{i,j=1,\dots,d}$ is called *turbulent diffusivity* of the tracer. It is generally believed that, if the field is stationary and sufficiently strongly mixing, then the central limit theorem holds, see e.g. Arnold (1964), Kraichnan (1970) and Taylor (1923).

It is expected, see Majda and Kramer (1999), Arnold (1964) and Kraichnan (1970), that both the LLN, with $v_* = 0$, and the CLT for the tracer trajectory hold when the velocity field is zero mean, Gaussian, incompressible and its covariance matrix $R(t, x) = [R_{pq}(t, x)]_{p,q=1,\dots,d}$, given by

$$R_{pq}(t, x) := \mathbb{E}[V_p(t, x)V_q(0, 0)], \quad p, q = 1, \dots, d, \ (t, x) \in \mathbb{R}^{1+d},$$

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exponentially decays in time, i.e. there exists $C > 0$ such, that

$$\sum_{p,q=1}^{d} |R_{pq}(t,x)| \le Ce^{-|t|/C}, \text{ for all } (t,x) \in \mathbb{R}^{1+d}. \tag{1.2}$$

The CLT has been established in Komorowski and Papanicolaou (1997), in the case of $T-$dependent fields, i.e. those for which exists $T > 0$ such that $R(t,x) = 0,\ |t| > T,\ x \in \mathbb{R}^d$.

In case when the vector field $V(t,x)$ is Markovian (not necessarily Gaussian) and satisfies the spectral gap condition, the CLT has been established in Fannjiang and Komorowski (1999), Theorem A, see also Komorowski et al. (2012), Carmona and Xu (1996) and Koralov (1999). In the Gaussian case when the covariance matrix is of the form

$$R_{pq}(t,x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi-\gamma(\xi)|t|}\hat{R}_{pq}(d\xi), \quad p,q = 1,\dots,d,\ (t,x) \in \mathbb{R}^{1+d}, \tag{1.3}$$

where both $\gamma(\cdot)$ and non-negative Hermitian matrix valued measure $\hat{R}(\cdot) = [\hat{R}_{pq}(\cdot)]$ are even (because the field is real), then the field is Markovian. It can be shown, see Chapter 12 of Komorowski et al. (2012), that the spectral gap condition holds, provided there exists $\gamma_0 > 0$ such that

$$\gamma(\xi) \ge \gamma_0, \quad \xi \in \mathbb{R}^d. \tag{1.4}$$

Until now the CLT has not been proved in such a generality for fields satisfying (1.2). We would like to perform a step in this direction. In this paper we consider non-Markovian fields, where the exponential factor in (1.3) is replaced by a function $h : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$ i.e. the covariance matrix is defined as follows

$$R_{pq}(t,x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} h(|t|,\xi)\hat{R}_{pq}(\xi)d\xi, \qquad p,q = 1,\dots,d,\ (t,x) \in \mathbb{R}^{1+d}, \tag{1.5}$$

where $\hat{R}_{pq}(\xi)$ is a density of $\hat{R}_{pq}(\cdot)$.

We show in Proposition 2.1 that the function h is non-negative definite iff $R(t,x) = [R_{pq}(t,x)]$ is non-negative definite. Therefore, the largest (in the sense of inclusion) set of functions h in (1.5) which can be examined is the class of non-negative definite functions.

In this paper, we study a smaller family of functions, namely we assume that h is *completely monotone in the sense of Bernstein*, see (1.7).

Let us denote by $\hat{r}$ the *power–energy spectrum*. It is a scalar non-negative, integrable function given by formula

$$\hat{r}(\xi) := \mathrm{tr}\hat{R}(\xi), \quad \xi \in \mathbb{R}^d, \tag{1.6}$$

where tr is the trace. Let $\mathfrak{r}(d\xi) := \hat{r}(\xi)d\xi$.

The main result of the paper, see Theorem 2.1, is the CLT for the trajectory of a tracer moving in a field whose covariance matrices are given by (1.5), where the function h is *completely monotone* i.e.

$$h \in C^\infty(0,+\infty) \cap C[0,+\infty), \quad (-1)^n h^{(n)}(t,\xi) \ge 0,\ t > 0,\ \mathfrak{r} \text{ a.e. } \xi,\ n = 0,1,\dots$$

and satisfies (1.8). From the Bernstein Theorem (Lax, 2002 Theorem 3., p. 138) we know that

$$h(t,\xi) = \int_0^{+\infty} e^{-\lambda t}\mu(\xi,d\lambda), \qquad t \in [0,+\infty),\ \mathfrak{r} \text{ a.e. } \xi, \tag{1.7}$$

where $\mu(\xi,\cdot)$ is a non-negative, finite Borel measure on $[0,+\infty)$ for $\mathfrak{r}$ a.e. ξ. For example from Schilling et al. (0000), Lemma 4.5, we know, that all completely monotone functions are non-negative definite. We assume that there exists $\lambda_0 > 0$ such that

$$\mathrm{supp}\ \mu(\xi,\cdot) \subset [\lambda_0,+\infty), \quad \mathfrak{r} \text{ a.e. } \xi. \tag{1.8}$$

Observe that this assumption implies (1.2). Note also that when $\mu(\xi,d\lambda) = \delta_{\gamma(\xi)}(d\lambda)$, we recover the Markovian case, see (1.3). In this way we generalize results from Fannjiang and Komorowski (1999), Koralov (1999) and Carmona and Xu (1996).

In Section 3.1 we show (see (3.7)) an example of a covariance matrix which is of the form (1.5) but not of the form (1.3).

We prove Theorem 2.1 in Section 3 by embedding the field $V(t,x)$ into a larger space where we add one dimension and one argument i.e. a space of $d+1$ dimensional fields $\tilde{V}(t,x,y)$. We define a field $\tilde{V}(t,x,y)$ in such a way that the field $(\tilde{V}(t,x,0))$ has the same distribution as the field $(V(t,x),0)$. The process $\tilde{V}(t,\cdot,\cdot)$ has the Markov property in appropriate function space. This process also has the spectral gap property. At the end we use Theorem 12.13 from Komorowski et al. (2012).

In Section 4 we show the proof of Proposition 2.1.

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