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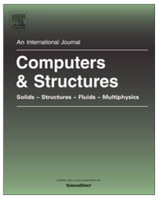

Simple and extensible plate and shell finite element models through automatic code generation tools

Jack S. Hale [a], Matteo Brunetti [b,c], Stéphane P.A. Bordas [a,d], Corrado Maurini [c,*]

[a] *Institute of Computational Engineering, University of Luxembourg, Luxembourg*
[b] *Dipartimento di Ingegneria Strutturale e Geotecnica, Sapienza Universitá di Roma, Italy*
[c] *Sorbonne Université, CNRS, Institut Jean Le Rond d'Alembert, UMR 7190, F-75005 Paris, France*
[d] *School of Engineering, Cardiff University, United Kingdom*

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ABSTRACT

A large number of advanced finite element shell formulations have been developed, but their adoption is hindered by complexities of transforming mathematical formulations into computer code. Furthermore, it is often not straightforward to adapt existing implementations to emerging frontier problems in thin structural mechanics including nonlinear material behaviour, complex microstructures, multi-physical couplings, or active materials. We show that by using a high-level mathematical modelling strategy and automatic code generation tools, a wide range of advanced plate and shell finite element models can be generated easily and efficiently, including: the linear and non-linear geometrically exact Naghdi shell models, the Marguerre-von Kármán shallow shell model, and the Reissner-Mindlin plate model. To solve shear and membrane-locking issues, we use: a novel re-interpretation of the Mixed Interpolation of Tensorial Component (MITC) procedure as a mixed-hybridisable finite element method, and a high polynomial order Partial Selective Reduced Integration (PSRI) method. The effectiveness of these approaches and the ease of writing solvers is illustrated through a large set of verification tests and demo codes, collected in an open-source library, FEniCS-Shells, that extends the FEniCS Project finite element problem solving environment.

1. Introduction

Plates and shells are solids occupying a spatial domain with one dimension, the thickness, much smaller than the others. This implies the possibility of experiencing large changes of shape even with small material deformations and an approximately linear elastic material behaviour. Their study has received renewed attention in the last decade because of their unique nonlinear behaviour, mainly caused by geometric effects [12], leading to d-cone singularities and crumpling [33], fracture [84], or multistability [89,90]. Examples of modern application fields in engineering include shape control through active materials [52,43], stretchable electronics [81], soft robotics [63], and thin nano-structures e.g. graphene sheets and nanotubes [11].

Effective models for plates and shells are two-dimensional, the through-the-thickness kinematics being described by including a suitable microstructure in the model. Their mathematical modelling leads to set of non-linear partial differential equations (PDEs) defined on a two-dimensional manifold embedded in three-dimensional space. The resulting PDEs can be discretised using a variety of numerical techniques, e.g. finite element methods (FEM) [15,76], isogeometric analysis [49,20], and meshfree methods [42]. FEM-based discretisations of the plate and shell models are available in both commercial (see e.g. Abaqus [2], AceFem [56] and Adina [14] and Ls-Dyna) and open-source (e.g. GetFem++ [77], IgaFem [75], Code_Aster [34], Mat-Fem [1] and Elmer [78]) software packages. However, formulating the mathematical models and implementing finite element solvers for custom plates and shells models remains a complex, highly technical, and time-consuming task, requiring advanced knowledge in differential geometry, numerical analysis, and mechanics. Shell and plate finite element models are regarded as very advanced topics in structural engineering and applied mathematics curricula, and rarely are graduate students trained in their implementation.

Our work aims to overcome some of the above difficulties by using modern automatic code generation tools and suitable mathematical abstractions. We present easily extensible and customisable methods for the implementation of finite element solvers for various models of thin structures, ranging from linear plates to non-linear shells. We leverage the tools offered by the finite element library FEniCS Project [4], in particular its Unified Form

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[*] Corresponding author.
E-mail address: corrado.maurini@upmc.fr (C. Maurini).

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Language (UFL) [5] and the associated FEniCS Form Compiler (FFC) [50], to allow users to straightforwardly formulate complex custom nonlinear shell models with less than one hundred lines of Python code and syntax that closely mirrors the mathematical abstractions of the variational formulation. We exploit the symbolic processing capabilities of UFL to specify the potential energy functionals of various thin structural models in a high-level way and select suitable locking-free finite element spaces for their discretisation. Custom constitutive models are straightforward to implement. Consistent linearisation of the potential energy functional is performed automatically using the symbolic differentiation tools in UFL [5]. These symbolic expressions are compiled automatically to finite element code using FFC [50]. The differential geometry concepts [31] are expressed directly in a very simple and expressive format, which we believe is of great pedagogical value. We mimic the geometrically-exact shell model [31], where the reference configuration is described by a mapping from $\mathbb{R}^2$ to a surface embedded in Euclidean space. Although we do not discuss here the implementation of specific preconditioners, our finite element solvers can run without modification on high-performance computing architectures using MPI and PETSc [13] allowing large-scale calculations to be performed. State-of-the-art linear and non-linear solution strategies are immediately available through PETSc [13]. These aspects sum up to a unique approach that we believe will be of value to researchers, students, and practitioners working on frontier problems in the mechanics of thin structures. The outcome of our work is distributed in the form of an open-source (LGPLv3) Python library, FEniCS-Shells [39], collecting an implementation of the models and discretisation techniques presented in this paper and including a large set of examples and documented demos.

We consider here several *shearable* plate and shell models, including linear plates (Reissner-Mindlin model [80,70]), linear and nonlinear shells (Nagdhi model [72,71]), and weakly nonlinear shallow shells (Marguerre-von Kármán model [67]). For these structural models, the discretisation strategy is particularly important, because shear and membrane locking can lead to an unacceptably slow convergence rate of the finite element solution with the mesh size [28]. After a deep analysis of the methods available in the literature and the possible software tools, we selected two main discretisation techniques: the popular Mixed Interpolation of Tensorial Components (MITC) approach [15,16,35] and a high polynomial-order Partial Selective Reduced Integration (PSRI) approach [9], both of which can be applied in a uniform manner across user-defined models.

Both the MITC and PSRI approaches presented here include some original aspects with respect to the formulations previously introduced in the literature. In order to fit within the abstractions of UFL and the associated code generation tools, MITC is reformulated as a mixed hybridisable variational form with element-wise local projection. The PSRI technique proposed in [9] is extended to nonlinear shells with a modified reduced integration rule and optimised weighting factor for the energy splitting. To provide an efficient parallel implementation of the MITC technique, we also extended the FEniCS automatic assembly tools to include the possibility of eliminating local degrees of freedom through static condensation, a feature that could be potentially ported to the FEniCS library itself in the near future.

To verify the convergence of the adopted MITC and PSRI discretisation techniques and illustrate the capability of the FEniCS-Shells library, we report a large set of numerical benchmark and verification tests. Beside many classical examples, we propose a novel numerical benchmark problem based on the exact solution of Mansfield [64] for the thermal buckling of a lenticular plate. We believe that this latter example can usefully complement the traditional gallery of verification tests for plates and shells.

The paper is organised as follows. First, we give an overview of the structural models considered in our work, presenting the research of their equilibrium configurations as an energy minimisation problem (Section 2). Hence we discuss the mathematics behind their finite element discretisation and some details about their practical implementation (Section 3). Finally we show some comprehensive verification examples demonstrating what is possible with our approach and the performance of the proposed methods (Section 4). Three detailed documented demos are included as supplementary material and several more are available in the online repository of the python library FEniCS-Shells, companion to this paper.

A permanent DOI [39] has been created with links to the latest code and documentation. Table 1 summarises the mathematical models and the numerical methods presented in each of the documented demos currently available online. The two adopted discretisation techniques, MITC and PSRI, can in principle be applied indifferently to the different models. We mainly suggest to use MITC for curing shear-locking in linear plates and PSRI in nonlinear shells, because the specific MITC formulation that we implemented turns out to be less effective for membrane locking (see e.g. the examples of Section 4.3). However all the combinations of models and discretisations are in principle possible and the users can easily adapt the provided demos to new models and implement variants of the available discretisations. We suggest the reader consult the demos to have an overview of the main capabilities and features of FEniCS-Shells, and in particular the *Clamped Reissner-Mindlin plate under uniform load* demo for linear plates, the *Buckling of a heated von-Kármán plate* for weakly nonlinear plate models, and the *Clamped semi-cylindrical Naghdi shell under point load* demo for fully nonlinear shells. FEniCS-Shells follows the *Best Practices for Scientific Computing* [96] as closely as possible, including using version control, continuous integration and testing, repeatable computing environments [40] and providing thorough documentation.

Table 1

Structural models and discretisation techniques presented in main documented demos provided in FEniCS-Shells [39]. All the demos adopt shareable shell theory, except the *Clamped Kirchhoff-Love plate* which is based on the discontinuous Galerkin discretisation technique presented in [95,74], not discussed in the present paper. All the combinations of models and discretisations are in principle possible, although we obtained better results using the proposed MITC implementation for linear plates and PSRI for linear and nonlinear shells.

Title	Model	Discretisation
Clamped Reissner-Mindlin plate under uniform load	Linear shearable plate	MITC
Simply supported Reissner-Mindlin plate	Linear shearable plate	MITC
Clamped Reissner-Mindlin plate with MITC7	Linear shearable plate	MITC
Clamped Kirchhoff–Love plate	Linear unshearable plate	DG
Buckling of a heated von Kármán plate	Shearable weakly nonlinear plate	PSRI
Non-linear Naghdi roll-up cantilever	Nonlinear shearable shell	MITC
Clamped semi-cylindrical Naghdi shell under point load	Nonlinear shearable shell	PSRI
Partly Clamped Hyperbolic Paraboloid	Linear shearable shell	PSRI

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