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Two criteria to check whether ideals are direct sums of cyclically presented modules ☆

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ABSTRACT

A famous theorem of commutative algebra due to I. M. Isaacs states that "if every prime ideal of R is principal, then every ideal of R is principal". Therefore, a natural question of this sort is "whether the same is true if one weakens this condition and studies rings in which ideals are direct sums of cyclically presented modules?" The goal of this paper is to answer this question in the case R is a commutative local ring. We obtain an analogue of Isaacs's theorem. In fact, we give two criteria to check whether every ideal of a commutative local ring R is a direct sum of cyclically presented modules, it suffices to test only the prime ideals or structure of the maximal ideal of R. As a consequence, we obtain: if R is a commutative local ring such that every prime ideal of R is a direct sum of cyclically presented R-modules, then R is a Noetherian ring. Finally, we describe the ideal structure of commutative local rings in which every ideal of R is a direct sum of cyclically presented R-modules.

1. Introduction

The study of prime ideals and maximal ideals of a ring R to specify the structure of all ideals of R has a long history in commutative algebra. For a synopsis of the history of this pursuit, going back to the works of I. M. Issacs, I. Kaplansky and I. S. Cohen. In particular, Issacs established the following theorem.

Result 1.1. (Attributed to I. M. Isaacs in [5, p. 8, Exercise 10].) For a commutative ring R, every prime ideal of R is principal if and only if every ideal of R is principal.

I. S. Cohen obtained in [4] the following theorem.

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Result 1.2. (Cohen [4]). For a commutative ring R, every prime ideal of R is finitely generated if and only if every ideal of R is finitely generated.

Also, I. Kaplansky proved in [6, Theorem 12.3] the following theorem.

Result 1.3. (Kaplansky [6, Theorem 12.3]). For a commutative Noetherian ring R, every maximal ideal of R is principal if and only if every ideal of R is principal.

The study of commutative local rings R in which all ideals of R are direct sums of cyclic modules was initiated by Behboodi et al. in [1], [2] and [3]. In particular, they established the following theorem.

Result 1.4. (see [2, Theorem 3.7]) Let $(R, \mathcal{M})$ be a commutative local ring. Then the following statements are equivalent:

(1) Every ideal of R is a direct sum of cyclic R-modules.
(2) There is an index set Λ and a set of elements $\{x, y\} \cup \{w_\lambda\}_{\lambda \in \Lambda} \subseteq R$ such that $\mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ with: each Rw_λ a simple R-module, $R/\mathrm{Ann}(x)$, $R/\mathrm{Ann}(y)$ principal ideal rings.
(3) Every ideal of R is a direct summand of a direct sum of cyclic R-modules.

A natural question arises. Instead of considering rings for which all ideals are direct sums of cyclic modules, we study rings in which all ideals are direct sums of cyclically presented modules. In this paper, we study commutative local rings in which every ideal is a direct sum of cyclically presented modules. In particular, we describe the ideal structure of such rings.

In [2, Example 4.3], Behboodi et al. gave an example showing that there exist commutative local rings R such that every prime ideal of R is a direct sum of cyclic R-modules, but some of the ideals of R are not direct sums of cyclic R-modules. In this direction, we give two criteria which are analogues of Isaacs's theorem. It is shown that to check whether every ideal of a commutative local ring R is a direct sum of cyclically presented modules, it suffices to test only the prime ideals or structure of the maximal ideal of R. In fact, the main results of this paper are the following.

Result 1.5 *(Main Theorem, see Theorem 2.16).* For a commutative local ring $(R, \mathcal{M})$, the following statements are equivalent:

(1) Every ideal of R is a direct sum of cyclically presented R-modules.
(2) Every ideal of R is a direct summand of a direct sum of cyclically presented R-modules.
(3) Every prime ideal of R is a direct sum of cyclically presented R-modules.
(4) Every prime ideal of R is a direct summand of a direct sum of cyclically presented R-modules.
(5) Either R is a principal ideal ring or $\mathcal{M} = Rx \oplus Ry$, where $x, y \notin \mathrm{Nil}(R)$, $R/\mathrm{Ann}(x)$ and $R/\mathrm{Ann}(y)$ are principal ideal rings.

Also, we show that commutative local rings whose ideals are direct sums of cyclically presented modules are Noetherian (Corollary 2.12). As a consequence, we obtain: if R is a uniserial ring, then R is a principal ideal ring if and only if every prime ideal of R is a direct sum of cyclically presented R-modules (Corollary 2.13). Finally, we describe the ideal structure of such rings (see Proposition 2.3 and Corollary 2.15).

Throughout the paper, R will denote a commutative ring with identity, $J(R)$ will denote its Jacobson radical and all modules will be assumed to be unitary. For a ring R we denote by $\mathrm{Spec}(R)$ and $\mathrm{Nil}(R)$ the set of prime ideals and the set of all nilpotent elements of R, respectively. A ring R is called *local* if R has a unique maximal ideal. In this paper, $(R, \mathcal{M})$ will be a local ring with maximal ideal $\mathcal{M}$. Also, an R-module

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