ARTICLE IN PRESS
JPAA:5902

Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

A cohomological point of view on gradings on algebras with multiplicative basis

L. Bemm, E.Z. Fornaroli, E.A. Santulo Jr. *

Departamento de Matemática, Universidade Estadual de Maringá, Avenida Colombo, 5790, Maringá, PR, 87020-900, Brazil

A R T I C L E   I N F O

Article history:
Received 4 September 2017
Received in revised form 7 February 2018
Available online xxxx
Communicated by A. Solotar

MSC:
16W50; 16E99

A B S T R A C T

In this paper we generalize the concepts of good and elementary gradings for an associative algebra A with a fixed multiplicative basis B. When the group G considered in the grading is abelian, we equip the set of good G-gradings of A with a structure of abelian group, which is denoted by $\mathcal{G}(B,G)$. Moreover, when A admits elementary G-gradings, we show the set $\mathcal{E}(B,G)$ of all elementary G-gradings of A is a subgroup of $\mathcal{G}(B,G)$. In this case, we introduce a cohomology for the pair (A,B) and we show that $\mathcal{G}(B,G)/\mathcal{E}(B,G)$ is isomorphic to the first cohomology group of (A,B) with coefficients in G.

1. Introduction

Let K be a field and let G be a multiplicative group. A G-grading on a K-algebra A is a decomposition of A into a direct sum of linear subspaces A_g, indexed by elements of G, such that $A_gA_h \subset A_{gh}$, for any $g,h \in G$. The elements of A_g are called *homogeneous of degree g*. In this case, we will denote the G-degree of a by $\|a\|$.

Introducing a convenient G-grading on an algebra A is a tool which may be useful to study specific properties of A in different contexts and, therefore, G-gradings of algebras are by themselves an important subject of study. For instance, $\mathbb{Z}_2$-gradings played a major role in Kemer's proof in [5] of Specht's conjecture. We are particularly interested in good and elementary gradings.

A good G-grading of the matrix algebra $M_n(K)$ is a grading where all unit matrices e_{ij} are G-homogenous. When there exists a map $f : \{1,2,\ldots,n\} \to G$ such that $\|e_{ij}\| = f(i)^{-1}f(j)$, the grading is said to be elementary. Good gradings appeared first in quotients of path algebras, although not with that name, in [3]. Elementary gradings on full matrix algebras were first considered, not with that name, in [1], where it is

* Corresponding author.
E-mail addresses: lbemm2@uem.br (L. Bemm), ezancanella@uem.br (E.Z. Fornaroli), easjunior@uem.br (E.A. Santulo).

https://doi.org/10.1016/j.jpaa.2018.04.019

Please cite this article in press as: L. Bemm et al., A cohomological point of view on gradings on algebras with multiplicative basis, J. Pure Appl. Algebra (2018), https://doi.org/10.1016/j.jpaa.2018.04.019

shown, in Proposition 2.1, that every good grading on $M_n(K)$ is elementary. Good and elementary gradings are usually defined in other "matricial" algebras like subalgebras of $M_n(K)$, matrix algebras over more general algebraic structures, incidence algebras, etc. One of our aims in the present paper is to generalize those concepts for a larger class of algebras.

A natural question that arises when one deals with good gradings and elementary gradings of an algebra is: "is every good G-grading of an algebra an elementary grading?" and, if the answer is "no", "how large is the part of the elementary G-gradings in all good G-gradings of an algebra A?". In case of the matrix algebra over a field or a more general algebraic structure, the answer to the first question is always "yes", and the second question is automatically answered. But, for other algebras, as incidence algebras of posets or subalgebras of a matrix algebra, there can exist good G-gradings which are not elementary, depending on the poset considered. This question was studied by Jones in [4], but not completely answered, and recently Price extended Jones's construction to incidence algebras over preorders (see [7] for more details).

In the present paper, when the group G is abelian, we identify the good G-gradings of a K-algebra A with an abelian group denoted by $\mathcal{G}(B, G)$, where B is a suitable basis of A. Via this identification, we show that the elementary G-gradings of A form a subgroup $\mathcal{E}(B, G)$ of $\mathcal{G}(B, G)$. Moreover, we introduce a cohomology for the pair (A, B), in such a way that $\mathcal{G}(B, G)/\mathcal{E}(B, G)$ is isomorphic to the first cohomology group, in this cohomology, with coefficients in G. Although not mentioning gradings, it is shown in [6] that every good G-grading on an incidence algebra of a pre-ordered set is elementary if and only if a suitable cohomology group is trivial. The cohomology constructed here for a wider class of algebras generalizes that one introduced by Nowicki.

The paper is organized as follows.

In Section 2, we define multiplicative and elementary basis for K-algebras and we consider G-graded K-algebras with a multiplicative basis B. We also introduce the concepts of B-good and B-elementary gradings and, when dealing with abelian gradings, we identify the set of all good G-gradings with an abelian group $\mathcal{G}(B, G)$ and the set of all elementary G-gradings is identified with a subgroup $\mathcal{E}(B, G)$ of $\mathcal{G}(B, G)$.

In Section 3, we introduce a cohomology for algebras having an elementary basis B.

In Section 4, we construct an injective group homomorphism from $\mathrm{Hom}(C_1, G)$ to G^B, where C_1 is the first chain group. We also verify that, through this homomorphism, $\mathcal{G}(B, G)$ is isomorphic to $\ker \delta^2$ and $\mathcal{E}(B, G)$ is isomorphic to $\mathrm{Im}\, \delta^1$, where δ^1 and δ^2 are, respectively, the first and second coboundary maps in the cohomology introduced in Section 3. So, we conclude that $\mathcal{G}(B, G)/\mathcal{E}(B, G)$ is isomorphic to the first cohomology group of A with coefficients in G.

Finally, in Section 5, we calculate explicitly the first cohomology group of some concrete algebras, namely: the Grassmann algebra, monoid algebras and incidence algebras. In the latter case, we obtain that every good grading of the algebra is elementary if and only if the first cohomology group of the poset of the algebra with coefficients in the abelian group G is trivial, which answers completely the problem investigated by Jones in [4].

Throughout the article, G is a multiplicative group, K is a field and A is an associative K-algebra with an identity element.

2. Good and elementary gradings

We say that a linear basis B for the algebra A is *multiplicative* if B is closed under the multiplication of A up to multiplication by a scalar, i.e., given $u, v \in B$, there exists $t \in B$ such that $uv = \lambda t$, for some $\lambda \in K$. So, there is a natural way to provide a semigroup structure to $B \cup \{0\}$, defining an operation $*$ in $B \cup \{0\}$ in the following way: $u * v = 0$ if $uv = 0$ and $u * v = t$ if $uv = \lambda t \neq 0$. Notice that if $uv \in B$, then $u * v = uv$.