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Cancellation of projective modules over polynomial extensions over a two-dimensional ring

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ABSTRACT

Let R be a commutative Noetherian ring of dimension two with $1/2 \in R$ and let $A = R[X_1, \cdots, X_n]$. Let P be a projective A-module of rank 2. In this article, we prove that P is cancellative if $\wedge^2(P) \oplus A$ is cancellative.

1. Introduction

Let R be a commutative Noetherian ring. A finitely generated projective R-module P is said to be cancellative if $P \oplus R^n \simeq Q \oplus R^n$ implies $P \simeq Q$. A classical result of Bass [1] asserts that every finitely generated projective R-module of rank $> \dim(R)$ is cancellative. This is the best possible result in general (since tangent bundle of real 2-sphere is stably trivial but not trivial). However, if R is an affine algebra of dimension n over an algebraically closed field, then Suslin [14] proved that every projective R-module of rank $\geq n$ is cancellative. If R is an affine algebra of dimension n over an infinite perfect C_1-field k such that $1/n! \in k$, then Suslin [13] proved that R^n is cancellative. Subsequently, Bhatwadekar [4] proved that every projective R-module of rank n is cancellative. If R is an affine algebra of dimension n over $\mathbb{Z}$, then Vaserstein [15, Corollary 18.1, Theorem 18.2] proved that R^n is cancellative. Subsequently, Mohan Kumar, Murthy and Roy [12] proved that every projective R-module of rank n is cancellative.

Remember that Bhatwadekar's proof [4] used Suslin's result [13] that R^n is cancellative. Similarly, the proof of Mohan Kumar, Murthy and Roy [12] used Vaserstein's result [15]. Hence, in view of the above results, people asked the following question.

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Question 1.1. Let R be a commutative Noetherian ring of dimension n. Assume that R^n is cancellative. Is every projective R-module of rank n cancellative?

In [3, Example 3.11], Bhatwadekar has given an example of a smooth real affine surface R such that R^2 is cancellative, but $K_R \oplus R$ is not cancellative, where K_R is the canonical module of R. Thus, the above question has negative answer in general. But in the same paper [3], Bhatwadekar proved the following result.

Let R be a commutative Noetherian ring of dimension 2 and P be a projective R-module of rank 2. If $\wedge^2(P) \oplus R$ is cancellative, then P is cancellative.

In this paper we prove the following result (3.8).

Theorem 1.2. *Let R be a commutative Noetherian ring of dimension 2 with $1/2 \in R$. Let P be a projective $A = R[X_1, \cdots, X_n]$-module of rank 2. Suppose that $\wedge^2(P) \oplus A$ is cancellative. Then P is cancellative.*

The above theorem is proved using the following result (3.3).

Proposition 1.3. *Let R be a commutative Noetherian ring of dimension 2 and $R \hookrightarrow S$ be a finite subintegral extension. Let P and Q be two projective $R[X_1, \cdots, X_n]$-modules of rank 2 such that $\wedge^2(P) \simeq \wedge^2(Q)$ and $P \otimes S[X_1, \cdots, X_n] \simeq Q \otimes S[X_1, \cdots, X_n]$. Let $\chi : \wedge^2(P) \xrightarrow{\sim} \wedge^2(Q)$ and $\theta : P \otimes S[X_1, \cdots, X_n] \xrightarrow{\sim} Q \otimes S[X_1, \cdots, X_n]$ be isomorphisms. Let 'bar' denote reduction modulo $(X_1, \cdots, X_n)$. Let $\tau : \overline{P} \xrightarrow{\sim} \overline{Q}$ be an isomorphism such that $\wedge^2(\tau) = \overline{\chi}$. Further assume that $\tau \otimes S = \overline{\theta}$. Then $P \simeq Q$.*

We derive an interesting known result (due to Wiemers) from (1.3) as an application (see 3.5).

Theorem 1.4. *Let R be a commutative Noetherian ring of dimension 2 with $1/2 \in R$. Suppose that all projective R-modules are cancellative, then all projective $R[X_1, \cdots, X_n]$-modules are also cancellative.*

2. Preliminaries

All the rings considered in this paper are assumed to be commutative Noetherian. By dimension of a ring R we mean its Krull dimension and it will be denoted by $\dim(R)$. Projective modules are finitely generated and of constant rank.

In this section we collect some definitions and results for later use. We start with the following definition.

Definition 2.1. An extension $R \hookrightarrow S$ of rings is called *subintegral* if: (1) it is integral, (2) the induced map $\operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(R)$ is bijective, and (3) for each $\mathfrak{P} \in \operatorname{Spec}(S)$ the induced field extension $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow S_{\mathfrak{P}}/\mathfrak{P}S_{\mathfrak{P}}$ is trivial, where $\mathfrak{p} = \mathfrak{P} \cap R$.

Remark 2.2. We now recall some important facts about subintegral extensions. The reader may refer to [16], [11] and [9] for further details.

(1) Let $R \hookrightarrow S$ be a subintegral extension. Then $R_{red} \hookrightarrow S_{red}$ is also subintegral.
(2) Let $R \hookrightarrow S$ be an extension of rings and $R \hookrightarrow R'$ be a faithfully flat extension. Write $S' = S \otimes_R R'$. Then $R \hookrightarrow S$ is subintegral if and only if $R' \hookrightarrow S'$ is subintegral. In particular, if $R \hookrightarrow S$ is subintegral then $R[X_1, \cdots, X_n] \hookrightarrow S[X_1, \cdots, X_n]$ is also subintegral.
(3) Let $R \hookrightarrow S$ be a finite ring extension with the same total ring of fractions. Let C be the conductor ideal of R in S. Then $\operatorname{ht}(C) \geq 1$. In particular, if $R \hookrightarrow S$ is a finite subintegral extension of reduced rings then $\operatorname{ht}(C) \geq 1$.