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A generalization of lifting non-proper tropical intersections

Xiang He

A R T I C L E I N F O

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A B S T R A C T

Let X and X' be closed subschemes of an algebraic torus T over a non-archimedean field. We prove the rational equivalence as tropical cycles in the sense of [11, §2] between the tropicalization of the intersection product $X \cdot X'$ and the stable intersection $\mathrm{trop}(X) \cdot \mathrm{trop}(X')$, when restricted to (the inverse image under the tropicalization map of) a connected component C of $\mathrm{trop}(X) \cap \mathrm{trop}(X')$. This requires possibly passing to a (partial) compactification of T with respect to a suitable fan. We define the compactified stable intersection in a toric tropical variety, and check that this definition is compatible with the intersection product in [11, §2]. As a result we get a numerical equivalence between $\overline{X} \cdot \overline{X}'|_{\overline{C}}$ and $\overline{\mathrm{trop}(X) \cdot \mathrm{trop}(X')|_C}$ via the compactified stable intersection, where the closures are taken inside the compactifications of T and $\mathbb{R}^n$. In particular, when X and X' have complementary codimensions, this equivalence generalizes [15, Theorem 6.4], in the sense that $X \cap X'$ is allowed to be of positive dimension. Moreover, if $\overline{X} \cap \overline{X}'$ has finitely many points which tropicalize to $\overline{C}$, we prove a similar equation as in [15, Theorem 6.4] when the ambient space is a reduced subscheme of T (instead of T itself).

1. Introduction

Let K be an algebraically closed field with a valuation $\mathrm{val}\colon K \to \mathbb{R}$. Let T be an algebraic torus of dimension n over K. There is a tropicalization map $\mathrm{trop}\colon T \to \mathbb{R}^n$ defined by taking the valuation of every coordinate. Under this map, the image of a pure dimensional subscheme X of T is a balanced polyhedral complex of the same dimension, which is denoted by $\mathrm{trop}(X)$. It is natural to consider under what conditions does the intersection commute with tropicalization, namely, given subschemes $X, X' \subseteq T$ when do we have $\mathrm{trop}(X \cap X) = \mathrm{trop}(X) \cap \mathrm{trop}(X')$. This reduces to a lifting problem since we always have $\mathrm{trop}(X \cap X') \subseteq \mathrm{trop}(X) \cap \mathrm{trop}(X')$. Results in this direction have been applied in [5], studying the connection between the theta characteristics of a K_4-curve and the theta characteristics of its minimal skeleton; and in [8], showing that the tropicalization of an irreducible subvariety of an algebraic torus is connected through codimension one; and in [6], discussing the lifting of divisors on a chain of loops as the skeleton of a smooth projective curve, etc.

E-mail address: xianghe@math.ucdavis.edu.

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When $\operatorname{trop}(X)$ intersects $\operatorname{trop}(X')$ properly this problem is studied thoroughly by Osserman and Payne in [14]. They proved that $\operatorname{trop}(X \cap X') = \operatorname{trop}(X) \cap \operatorname{trop}(X')$, which generalizes a well-known result [3, Lemma 3.2] concerning the lifting when $\operatorname{trop}(X)$ and $\operatorname{trop}(X')$ intersect transversely. Moreover, they gave a lifting formula for the intersection multiplicity of $\operatorname{trop}(X) \cdot \operatorname{trop}(X')$ along a maximal face of $\operatorname{trop}(X) \cap \operatorname{trop}(X')$, where the ambient space is a closed subscheme of T (instead of T itself). See [14, §5] for details.

The commutativity does not hold when $\operatorname{trop}(X) \cap \operatorname{trop}(X')$ is nonproper. For example one can take hyperplanes $X = \{x = 1\}$ and $X' = \{x = 1 + a\}$ where $\operatorname{val}(a) > 0$, then X and X' have empty intersection and same tropicalizations. However, one can still ask about the connections between the intersection cycles $X \cdot X'$ and $\operatorname{trop}(X) \cdot \operatorname{trop}(X')$. As an example, assume K is nonarchimedean, Morrison [12] proved that when X and X' are plane curves that intersect properly, the tropicalization of the intersection cycle $\operatorname{trop}(X \cdot X')$ is rationally equivalent to $\operatorname{trop}(X) \cdot \operatorname{trop}(X')$ as divisors on the (possibly degenerated) tropical curve $\operatorname{trop}(X) \cap \operatorname{trop}(X')$. In the higher dimensional case, Osserman and Rabinoff proved in [15, Theorem 6.4] that when X and X' are of complementary codimension, the number of points of $X \cap X'$, after a suitable compactification of the torus, that tropicalize to (the closure in the corresponding compactification of $\mathbb{R}^n$ of) a connected component C of $\operatorname{trop}(X) \cap \operatorname{trop}(X')$ is the same as the number of points in $\operatorname{trop}(X) \cdot \operatorname{trop}(X')$ supported on C, where both numbers are assumed to be finite and are counted with multiplicities.

In this paper we generalize the result of [15] in several directions. First assume X and X' do not necessarily intersect properly, hence the intersection multiplicity is not well defined. Instead of counting points of their intersection, we look at the refined intersection product $\overline{X} \cdot \overline{X}'$ on $\overline{X} \cap \overline{X}'$ which as a cycle class is represented by a formal sum of points supported on $\overline{X} \cap \overline{X}'$ ([10, §8]), where the closures are taken inside the toric variety $X(\Delta)$ associated to a certain unimodular fan Δ (hence $X(\Delta)$ is smooth). Restricting to the closure of a component of $\operatorname{trop}(X) \cap \operatorname{trop}(X')$ in the corresponding compactification, denoted by $N_{\mathbb{R}}(\Delta)$, of $\mathbb{R}^n$ we have:

Theorem 1.1. *Let X and X' be closed subschemes of T of complementary codimensions, C a connected component of* $\operatorname{trop}(X) \cap \operatorname{trop}(X')$. *Then there exists a fan Δ such that the degree of the subset of $\overline{X} \cdot \overline{X}'$ that tropicalizes to $\overline{C}$ is the same as that of* $\operatorname{trop}(X) \cdot \operatorname{trop}(X')$ *supported on C.*

In particular we are allowed to consider self-intersections of subschemes of T (see Example 4.4), which is not mentioned in [15]. The idea of proof is similar to [15, Theorem 6.4], namely, we show that the intersection cycle $\overline{X} \cdot \overline{X}'$ can be approached by the intersection of $\overline{X}$ and a perturbation of $\overline{X}'$, when restricted to a neighborhood of $\overline{C}$. Here by perturbation we mean $t\overline{X}'$ for some $t \in T$ with $\operatorname{val}(x_i(t))$ small enough. The argument requires passing to nonarchimedean analytic spaces. This case will be discussed in Section 4, see Theorem 4.3. Moreover, a sufficient condition for the fan Δ will be given.

Theorem 1.1 is easily generalized to multiple intersections (see Theorem 4.8) which plays an important role in our next approach of generalization, namely testing higher dimensional intersections. Let $i_{\overline{C}}\colon Z_{\overline{C}} \to \overline{X} \cap \overline{X}'$ be the inclusion of the union of irreducible components of $\overline{X} \cap \overline{X}'$ that tropicalize to $\overline{C}$; note that this is an open and closed subset inclusion. Assuming $\dim(X) + \dim(X')$ is greater than or equal to n, we prove that, after restricting to $\overline{C}$, we have $\operatorname{trop}(\overline{X} \cdot \overline{X}')$ rationally equivalent to the closure of $\operatorname{trop}(X) \cdot \operatorname{trop}(X')$ as tropical cycles in $N_{\mathbb{R}}(\Delta)$. Specifically, we have:

Theorem 1.2. *Let X and X' be closed subschemes of T of pure dimensions k and l, and C a connected component of* $\operatorname{trop}(X) \cap \operatorname{trop}(X')$. *Then there is a family of fans Δ such that*

$$[\operatorname{trop}(i_{\overline{C}}^*(\overline{X} \cdot \overline{X}'))] = [\overline{\operatorname{trop}(X) \cdot \operatorname{trop}(X')|_C}] \in A_{k+l-n}(N_{\mathbb{R}}(\Delta)).$$

See Section 3.1 or [11, §2] for the definitions of tropical cycles and rational equivalence on $N_{\mathbb{R}}(\Delta)$. In particular, rational equivalence preserves the degrees of zero cycles. In Section 3 we develop a compactified

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