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Weighted trapezoidal inequalities related to the area balance of a function with applications

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ABSTRACT

Some basic results in connection with the area balance function associated to a Lebesgue integrable function are obtained and then two new Fejér trapezoidal type inequalities are presented. Also some applications for random variable, trapezoidal formula and special means are given.

1. Introduction

In 1906, Fejér [8] obtained the following integral inequalities known in the literature as Fejér inequality:

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1}$$

where $f:[a,b]\to\mathbb{R}$ is convex and $g:[a,b]\to\mathbb{R}^+=[0,+\infty)$ is integrable and symmetric to $x=\frac{a+b}{2}\big(g(x)=g(a+b-x), \forall x\in [a,b]\big)$. For more inequalities in connection with (1), we refer the readers to [10–14] and references therein.

The Fejér trapezoidal type inequality means the estimation of the difference between the right and middle part of (1), which has been obtained in [3] by Hwang, as the following:

Theorem 1.1. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable mapping on I°, where $a, b\in I$ with $a<b$, and let g: $[a, b]\to[0,\infty)$ be continuous positive mapping and symmetric to $\frac{a+b}{2}$. If the mapping $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx-\int_a^b f(x)g(x)dx\right|\\ &\quad\le\frac{(b-a)}{4}\Big[\big|f'(a)\big|+\big|f'(b)\big|\Big]\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(x)dxdt.\end{aligned} \tag{2}$$

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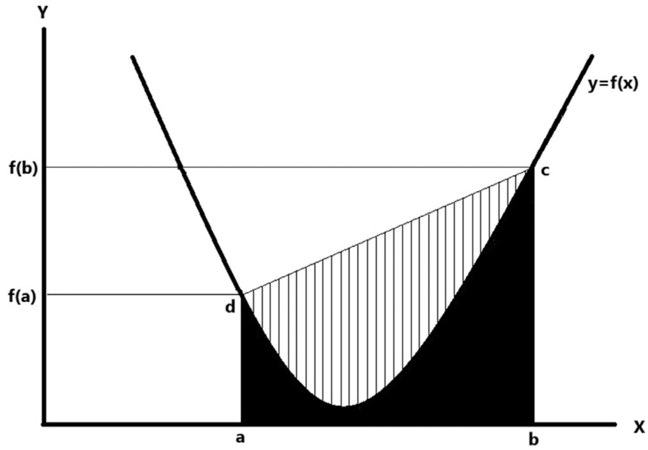

Fig. 1. Trapezoid type inequality.

We have chosen the name of Fejér trapezoidal type inequality for (2) because when $g \equiv 1$, it reduces to the following inequality (obtained in [6])

$$\left| \int_a^b f(x)dx - (b-a)\frac{f(a)+f(b)}{2} \right| \le \frac{1}{8}(b-a)^2\big(|f'(a)| + |f'(b)|\big), \tag{3}$$

which gives an estimation for the difference between the area of trapezoid *abcd* and the area under the graph of f as well (Fig. 1).

In 2006, the concept of h-convex functions related to the nonnegative real functions has been introduced in [15] by Varošanec. The class of h-convex functions is including a large class of nonnegative functions such as nonnegative convex functions, Godunova–Levin functions [9], s-convex functions in the second sense [2] and P-functions [7].

Definition 1.2. [15] Let $h : [0, 1] \to \mathbb{R}^+$ be a function such that $h \not\equiv 0$. We say that $f : I \to \mathbb{R}^+$ is a h-convex function, if for all $x, y \in I$, $\lambda \in [0, 1]$ we have

$$f\big(\lambda x + (1-\lambda)y\big) \le h(\lambda)f(x) + h(1-\lambda)f(y). \tag{4}$$

Obviously, if $h(t) = t$, then all non-negative convex functions belong to the class of h-convex functions. If we consider $h(t) = \frac{1}{t}$, $h(t) = t^s, s \in (0, 1]$, and $h(t) = 1$ in (4), respectively then we recapture the definitions Godunova–Levin functions, s-convex functions and P-functions, respectively. The Fejér inequality related to h-convex functions has been obtained in [1].

On the other hand, the *area balance function* associated to a Lebesgue integrable function has been introduced by Dragomir in [4,5] with the following backgrounds: For a Lebesgue integrable function $f : [a, b] \to \mathbb{C}$ and a number $x \in (a, b)$ there exists a question as how far the integral $\int_x^b f(t)dt$ is from the integral $\int_a^x f(t)dt$. If f is nonnegative and continuous on $[a, b]$, then the above question has the geometrical interpretation of comparing the area under the curve generated by f at the right of the point x with the area at the left of x. The point x is called a *median point*, if

$$\int_x^b f(t)dt = \int_a^x f(t)dt.$$

Due to the above geometrical interpretation, the area balance function associated to a Lebesgue integrable function $f : [a, b] \to \mathbb{C}$ has been defined as

$$AB_f(a, b, x) : [a, b] \to \mathbb{C}; \quad AB_f(a, b, x) = \frac{1}{2}\left[\int_x^b f(t)dt - \int_a^x f(t)dt\right],$$

or equivalently for any $t \in [0, 1]$ we have

$$AB_f(0, 1, t) = \frac{b-a}{2}\left[\int_0^t f\big(sa + (1-s)b\big)ds - \int_t^1 f\big(sa + (1-s)b\big)ds\right].$$

Utilizing the *cumulative function* notation $F : [a, b] \to \mathbb{C}$ given by

$$F(x) = \int_a^x f(t)dt,$$

then we observe that

$$AB_f(a, b, x) = \frac{1}{2}F(b) - F(x); \;\; x \in [a, b].$$