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An algebraic perspective on integer sparse recovery

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ABSTRACT

Compressed sensing is a relatively new mathematical paradigm that shows a small number of linear measurements are enough to efficiently reconstruct a large dimensional signal under the assumption the signal is *sparse*. Applications for this technology are ubiquitous, ranging from wireless communications to medical imaging, and there is now a solid foundation of mathematical theory and algorithms to robustly and efficiently reconstruct such signals. However, in many of these applications, the signals of interest do not only have a sparse representation, but have other structure such as lattice-valued coefficients. While there has been a small amount of work in this setting, it is still not very well understood how such extra information can be utilized during sampling and reconstruction. Here, we explore the problem of integer sparse reconstruction, lending insight into when this knowledge can be useful, and what types of sampling designs lead to robust reconstruction guarantees. We use a combination of combinatorial, probabilistic and number-theoretic methods to discuss existence and some constructions of such sensing matrices with concrete examples. We also prove sparse versions of Minkowski's Convex Body and Linear Forms theorems that exhibit some limitations of this framework.

1. Introduction

Initially motivated by a seemingly wasteful signal acquisition paradigm, compressed sensing has become a broad body of scientific work spanning across the disciplines of mathematics, computer science, statistics, and electrical engineering [13,16]. Described succinctly, the main goal of compressed sensing is sparse recovery – the robust *reconstruction* (or *decoding*) of a sparse signal from a small number of linear measurements. That is, given a signal $\boldsymbol{x} \in \mathbb{R}^d$, the goal is to accurately reconstruct $\boldsymbol{x}$ from its noisy measurements

$$\boldsymbol{b} = A\boldsymbol{x} + \boldsymbol{e} \in \mathbb{R}^m. \tag{1}$$

Here, A is an underdetermined matrix $A \in \mathbb{R}^{m \times d}$ ($m \ll d$), and $\boldsymbol{e} \in \mathbb{R}^m$ is a vector modeling noise in the system. Since the system is highly underdetermined, it is ill-posed until one imposes additional constraints, such as the signal $\boldsymbol{x}$ obeying a sparsity model. We say $\boldsymbol{x}$ is s-sparse when it has at most s nonzero entries:

$$\|\boldsymbol{x}\|_0 := |\mathrm{supp}(\boldsymbol{x})| = |\{i : x_i \neq 0\}| \leq s \ll d. \tag{2}$$

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Clearly, any matrix A that is one-to-one on $2s$-sparse signals will allow reconstruction in the noiseless case ($\boldsymbol{e} = 0$). However, compressed sensing seeks the ability to reconstruct *efficiently* and *robustly*; one needs a computationally feasible reconstruction method, and one that allows accurate reconstruction even in the presence of noise. Fortunately, there is now a large body of work showing such methods are possible even when m is only logarithmic in the ambient dimension, $m \approx s\log(d)$ [13,16]. Typical results rely on notions like incoherence, null-space property or the restricted isometry property [10], which are quantitative properties of the matrix A slightly stronger than simple injectivity. Under such assumptions, greedy (e.g. [5,19,23,27]) and optimization-based (e.g. [8,10]) approaches have been designed and analyzed that efficiently produce an estimation $\hat{\boldsymbol{x}}$ to an s-sparse signal $\boldsymbol{x} \in \mathbb{R}^d$ from its measurements $\boldsymbol{b} = A\boldsymbol{x} + \boldsymbol{e} \in \mathbb{R}^m$ that satisfies

$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}\| \lesssim \|\boldsymbol{e}\|, \tag{3}$$

where $\lesssim$ hides only constant factors and $\|\cdot\|$ will always denote the Euclidean norm. Although this body of work has blossomed into many other directions based on practical motivations, there is very little understanding about the role of lattice-valued signals in this paradigm. This is especially troubling given the abundance of applications in which the signal is known to have lattice-valued coefficients, such as in wireless communications [20], collaborative filtering [11], error correcting codes [9], and many more. Initial progress in this setting includes results for dense (not sparse) ± 1 signals [17], binary sparse signals [12,22], finite-alphabet sparse signals [24,26], and generalized lattice-valued signals [14]. The latter two categories are most relevant to our work; [24,26] propose modifications of the sphere decoder method that offer some empirical advantages but lack a rigorous theory. The recent work [14] provides some theoretical guarantees for the greedy method OMP [23] initialized with a pre-processing step, and also shows that rounding the result given by ℓ_1-minimization does not yield any improvements for many lattices. In this paper, our focus is not algorithmic but instead we aim to answer the questions (i) what kind of sensing matrices can be designed for lattice-valued sparse signals, and (ii) what are the limitations of the advantages one hopes to gain from knowledge that the signal is lattice-valued? Our perspective in this work is thus *algebraic*, and we leave algorithmic designs for such lattice-valued settings for future work. We view our contribution as the foundation of an algebraic framework for lattice-valued signal reconstruction, highlighting both the potential and the limitations.

2. Problem formulation and main results

Let $m < d$ and first consider the noiseless consistent underdetermined linear system

$$A\boldsymbol{x} = \boldsymbol{b}, \tag{4}$$

where A is an $m \times d$ real matrix and $\boldsymbol{b} \in \mathbb{R}^d$. Let us first consider when this system has a unique solution $\boldsymbol{x}$.

Notice that if $\boldsymbol{x}$ and $\boldsymbol{y}$ are two different such solution vectors, then

$$A(\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{0},$$

i.e. the difference vector $\boldsymbol{x} - \boldsymbol{y} \in \mathbb{N}(A)$, the null-space of the matrix A.

Let us write $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d \in \mathbb{R}^m$ for the column vectors of the matrix A. A vector $\boldsymbol{z} \in \mathbb{N}(A)$ if and only if

$$A\boldsymbol{z} = \sum_{i=1}^{d} z_i \boldsymbol{a}_i = \boldsymbol{0}, \tag{5}$$

i.e. if and only if $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d$ satisfy a linear relation with coefficients $z_1, \dots, z_n$. The uniqueness of solution to (4) (and hence our ability to decode the original signal) is equivalent to non-existence of nonzero solutions to (5).

Since $d > m$, such solutions to (5) must exist. On the other hand, if we add some appropriate restriction on the solution vectors in question, then perhaps there will be no solutions satisfying this restriction. In other words, the idea is to ensure uniqueness of decoded signal by restricting the original signal space. We can then formulate the following problem.

Problem 1. Define a restricted d-dimensional signal space $X \subseteq \mathbb{R}^d$ and an $m \times d$ matrix A with $m < d$ such that $A\boldsymbol{x} \neq 0$ for any $\boldsymbol{x} \in X$.

A commonly used restriction is sparsity, defined in (2). Now, while (5) has nonzero solutions, it may not have any nonzero s-sparse solutions for sufficiently small s. In addition to exploiting sparsity, one can also try taking advantage of another way of restricting the signal space. Specifically, instead of taking signals to lie over the field $\mathbb{R}$, we can restrict the coordinates to a smaller subfield of $\mathbb{R}$, for instance $\mathbb{Q}$. The idea here is that, while columns of our matrix A are linearly dependent over $\mathbb{R}$, they may still be linearly independent over a smaller subfield. For instance, we have the following trivial observation.

Lemma 2.1. *Let $K \subsetneq \mathbb{R}$ be a proper subfield of the field of real numbers, and let $\alpha_1, \dots, \alpha_d \in \mathbb{R}$ be linearly independent over K. Define $A = (\alpha_1 \ \dots \ \alpha_d)$ be a $1 \times d$ matrix, then the equation $A\boldsymbol{x} = 0$ has no solutions in K^d except for $\boldsymbol{x} = \boldsymbol{0}$.*

Of course, when (5) has no solutions, we can guarantee our system (4) has a unique solution $\boldsymbol{x}$ and can in theory will be able to decode successfully. However, for practical concerns we want to be able to tolerate noise in the system and decode robustly as in (1). Since in practice the noise $\boldsymbol{e}$ typically scales with the entries (or row or column norms) of A, we ask for the following two properties: