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Modelling temporal decay of aftershocks by a solution of the fractional reactive equation

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ABSTRACT

We propose a new analytical perspective to explain the behavior of the number of seismic events observed post an intense earthquake as time elapses, through the application of a fractional solution of the reactive equation. According to the results obtained, a double power law model shows the number density decay in several possible ways, among which is a particular case the modified version of Omori Law proposed by Utsu in 1961.

1. Introduction

The understanding of the behavior of the temporal frequency of a number of seismic events observed after an intense earthquake and its decay has been a question that has been explained by the work of F. Omori entitled "*On the aftershocks of earthquakes*" published in 1894 [11], and subsequently re-formulated in [13]. The last of these authors presented a form that shows the frequency of aftershocks per unit time interval $n(t)$ at time t, given by:

$$n(t) = \frac{k}{(C+t)^p} \tag{1}$$

with $k > 0$, $C > 0$ and $p > 0$. The parameter p controls the rate of decay of aftershocks over time, and has been found in [5] to have a value around 1.0. The parameter k corresponds to the productivity of the aftershocks, and the value of C is a time shift constant that relates to the aftershock rate in the early phase of the sequence of events, and the determination of its value has been the focus of several studies, as in [3,4] and [6], where in the latter, by evaluating the efficiency of various models in decribing the time decay of aftershock rate, the authors demonstrated that the modified Omori Model with C kept fixed to zero represents the better choise for the modeling and forecasting of simple sequence behavior in California and Italy. In theses cases they affirm that the parameter C in the Omori equation, does not represent a general feature of the aftershock rate decay. They pointed out it is mainly an artifice induced by catalog incompleteness in the first times after the main shock. But the Utsu-Omori form has also been questioned, and other forms have been proposed more recently. In [10] pointed out that all aftershock sequences defined by three declustering techniques in California and Taiwan, are best described by a stretched exponential than by a power law. They say that a stretched exponential function describes most relaxation data observed in Nature, so it is considered an universal property of relaxing systems, and compared to the pure exponential decay, it indicates that the decay rate is not constant but decreases with time as $t^{\beta-1}$ and its behavior is sensitive to the method from which aftershocks are defined.

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On the other hand, from the theoretical perspective Omori type-law has been treated to a lesser extent (e.g. [1], [12]), so that to support in this line, we present an application of the reactive equation from the perspective of fractional calculus, which constitute a powerfull framework that has already been applied by some authors to the study of earthquakes (see [26] to find other applications). In [7] have been applied fractional derivatives to seismic analysis of base-isolated models, showing that the fractional derivative model is superior to the conventional model in predicting the peak response for an earthquake simulated test. Also, in [2], by considering earthquakes as complex systems showed that fractional model is very efficient for earthquake modeling. In [15] was performed a Multi Dimensional Scaling (MDS) analysis to understand the global behaviour of earthquakes, and so visualize the similarities among Earth's seismic regions. In this way, MDS turned out to be a useful visual representation of the complex relationships present among seismic events, which are not perceived on traditional maps.

2. The model

2.1. Construction

A problem concerning reacting particles has been treated several times by Mathai A.M. and Haubold H.J. (see, for example, [9]), where for the *ith* particle the evolution of the time dependent number density $N_i(t)$ is written as

$$\frac{dN_i(t)}{dt} = -c\,N_i(t), \tag{2}$$

with $c > 0$ and $N_i(t=0) = N_0^{(i)}$. Given the situation where some produced particles are destroyed, we have that the c coefficient represents the corresponding residual rate of change. About that, we are interested in the case when the destruction rate dominates.

In this stage, we propose to relate $N_i(t)$ with the number density of seismic events, independent of their magnitude, observed after an earthquake in a certain geographic region, with rate of decrease given by c. Now, we will shape our seismic decay model from $N_i(t)$ through expression (2).

Just as can be seen in the references mentioned at the beginning of this section, a fractional integration is performing instead of the full integral, which will allow us to address non-exponential properties of decay process. Hence, dropping i, we have

$$N(t) - N_0 = -c^{\nu}\, {}_0D_t^{-\nu} N(t), \tag{3}$$

with $\nu > 0$ the fractional order of the integral applied and the constant N_0 is the initial condition for the number density on $t = 0$. There c was replaced by c^{ν} for physical reasons.

${}_0D_t^{-\nu}$ is the fractional integral operator of Riemann–Liouville, which in general is given by

$${}_aD_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-u)^{\nu-1} f(u)du. \tag{4}$$

It represents a generalization for noninteger order of the Cauchy's formula, which is an iterated integral that may be expressed as a weighted single integral with a weight function (it can be seen in [21]). Applying the Laplace transform in (3), using the property

$$\mathcal{L}\left[{}_0D_t^{-\nu} N(t)\right](s) = s^{-\nu} N(s) \tag{5}$$

it is possible to obtain that (we stress that the details can be seen in the aforementioned references)

$$N(t) = N_0 \sum_{j=0}^{\infty} \frac{(-1)^j [(ct)^{\nu}]^j}{\Gamma(1+j\nu)}, \tag{6}$$

which can be written as

$$N(t) = N_0\, E_{\nu}(-c^{\nu} t^{\nu}). \tag{7}$$

This is a solution for $N(t)$ in terms of the Mittag–Leffler function , whose general form $E^{\gamma}_{\alpha\beta}(z)$ was defined by [24]:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{(\gamma)_j\, z^j}{j!\,\Gamma(\beta+\alpha j)} \tag{8}$$

where for the real part the parameters $\beta > 0$, $\alpha > 0$ and $\gamma \neq 0$, where we have $(\gamma)_j = \gamma(\gamma+1)\cdots(\gamma+j-1)$ and $(\gamma)_0 = 1$. So, (7) contains its original form introduced in [25], wich is obtained with $\gamma = 1$ and $\beta = 1$.

The Mittag–Leffler function arises naturally in the solution of fractional order integral equations or fractional order differential equations, providing deviations from the exponential behavior of the observed physical phenomena. This fact, and other interesting properties, have generated a great interest of physicists in recent years. Evidence of this and its analytical importance can be seen in [14].