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Ordinary differential equations

Sharp uniqueness conditions for one-dimensional, autonomous ordinary differential equations

Conditions fines d'unicité pour les équations différentielles ordinaires autonomes en dimension 1

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A R T I C L E I N F O

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A B S T R A C T

We give two conditions that are necessary and sufficient for the uniqueness of Filippov solutions to scalar, autonomous ordinary differential equations with discontinuous velocity fields. When only one of the two conditions is satisfied, we give a natural selection criterion that guarantees uniqueness of the solution.

R É S U M É

Nous donnons deux conditions nécessaires et suffisantes pour l'unicité des solutions de Filippov des équations différentielles ordinaires autonomes scalaires, avec champs de vitesse discontinus. Lorsqu'une seule de ces deux conditions est satisfaite, nous donnons un critère naturel sélectionnant une unique solution.

1. Introduction and statement of the theorem

The purpose of this paper is to derive necessary and sufficient conditions for the uniqueness of Filippov solutions to the scalar, autonomous ordinary differential equation (ODE)

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}t}(t) = b(X(t)) & \text{for } t > 0 \\ X(0) = x_0 \end{cases} \tag{1}$$

where $b : \mathbb{R} \to \mathbb{R}$ is Borel measurable and locally bounded, and $x_0 \in \mathbb{R}$. If b is continuous then the sense in which (1) holds is classical: $X : [0, \infty) \to \mathbb{R}$ is absolutely continuous and $\frac{\mathrm{d}}{\mathrm{d}t}X(t) = b(X(t))$ holds for almost every $t > 0$. It was shown

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by Binding [3] that the solution is unique *if and only if* b satisfies the so-called Osgood condition at all zeroes of b (see below). For instance, any Lipschitz continuous b satisfies Osgood's condition. For a general reference on the uniqueness and non-uniqueness of ODEs, see [1].

If b is merely measurable, say, $b \in L^\infty(\mathbb{R})$, then the interpretation of (1) is more subtle, and choosing a different representative in the equivalence class of b can lead to very different solutions. For instance, redefining the constant velocity field $b(x) \equiv 1$ at a single point, $b(x_0) = 0$, yields both the solutions $X(t) \equiv x_0$ and $X(t) = x_0 + t$. Several authors have analyzed possible modifications of b on negligible sets in order to ensure the existence of a classical solution, see, e.g., [7,4] and the references therein. The concept of *Filippov flows* or *Filippov solutions* to (1) provides an alternative solution to this issue by choosing a canonical representation of the velocity field. More precisely, the differential equation (1) is replaced by a differential *inclusion* where the right-hand side contains information on the behavior of b in an infinitesimal neighborhood of $X(t)$. Filippov [6] showed that there exists a Filippov solution to (1) under very mild conditions on b, for instance if $b \in L^\infty(\mathbb{R})$ or, for local existence, $b \in L^\infty_{\mathrm{loc}}(\mathbb{R})$.

In Section 1.1, we provide the definition of Filippov solutions and, in Section 1.2, we describe the essential Osgood criterion. The main theorem of this paper, stated in Section 1.3, gives necessary and sufficient conditions for the uniqueness of Filippov solutions to (1). As a corollary, we define a class of functions $\tilde{b} : \mathbb{R} \to \mathbb{R}$, for which the corresponding ODE all have the same unique, *classical* solution. Section 2 contains the proof of the Theorem and its Corollary, while Section 3 lists some examples.

1.1. *Set-valued functions and Filippov solutions*

We say that an absolutely continuous function $X : [0, T) \to \mathbb{R}$ is a *Filippov solution* to (1) if $X(0) = x_0$ and

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) \in K[b](X(t)) \qquad \text{for a.e. } t \in (0, T)$$

(see [6]). Here, the set-valued function $K[b]$ is defined as

$$K[b](x) := \bigcap_{\delta>0} \bigcap_{\substack{N\subset\mathbb{R}\\ |N|=0}} \overline{\mathrm{conv}}\big(b\big(B_\delta(x) \setminus N\big)\big)$$

where $B_\delta(x)$ is the open ball around x with radius δ, and $\overline{\mathrm{conv}}(A)$ is the smallest closed, convex set containing A. The intersection is taken over all Lebesgue measurable sets $N \subset \mathbb{R}$ with one-dimensional Lebesgue measure $|N| = 0$. In a similar vein we define the essential upper and lower bounds of b at x as

$$\begin{aligned} m[b](x) &:= \min\big(K[b](x)\big) = \lim_{\delta\to 0} \operatorname*{ess\,inf}_{x'\in B_\delta(x)} b(x'), \\ M[b](x) &:= \max\big(K[b](x)\big) = \lim_{\delta\to 0} \operatorname*{ess\,sup}_{x'\in B_\delta(x)} b(x'). \end{aligned} \tag{2}$$

We will say that b is continuous at a point x if the set $K[b](x)$ contains a single point, otherwise we say that b is discontinuous at x. It is evident that this coincides with the usual definition of continuity at a point, possibly after redefining b on a negligible set.

We list below some properties that are straightforward to check (see also [2,5]):

(i) $K[b]$ is upper hemicontinuous;
(ii) if $0 \notin K[b](x)$ for some $x \in \mathbb{R}$ then there is a neighborhood U of x such that $0 \notin K[b](y)$ for every $y \in U$;
(iii) $m[b]$ and $M[b]$ are lower and upper semi-continuous, respectively;
(iv) the set of discontinuities of b coincides with the measurable set $\{x \;:\; m[b](x) < M[b](x)\}$.

1.2. *The Osgood condition*

The classical uniqueness result for ODEs requires Lipschitz continuity of the velocity field b. In 1898, Osgood relaxed this condition to mere continuity of b, along with an integrability condition on its reciprocal [8]. We recall the main idea of Osgood's condition here. We will call a function $g : (-\delta_0, \delta_0) \to [0, \infty)$ an *Osgood function* if it is nonnegative, Borel measurable, and satisfies:

$$\int_{-\delta}^{0} g(z)^{-1}\mathrm{d}z = +\infty, \qquad \int_{0}^{\delta} g(z)^{-1}\mathrm{d}z = +\infty \qquad \forall\, \delta \in (0, \delta_0). \tag{3}$$

Lemma 1 *(Osgood lemma). Let $g : (-\delta_0, \delta_0) \to [0, \infty)$ be an Osgood function and let $u : [0, T) \to (-\delta_0, \delta_0)$ be an absolutely continuous function satisfying $u(0) = 0$ and*

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