Cognition 149 (2016) 11–17

Contents lists available at ScienceDirect

Cognition

journal homepage: www.elsevier.com/locate/COGNIT

Brief article

Free versus anchored numerical estimation: A unified approach

John E. Opfer [a,*], Clarissa A. Thompson [b,1], Dan Kim [a]

[a] The Ohio State University, 245 Psychology Building, Columbus, OH 43210, USA
[b] University of Oklahoma, Dale Hall Tower, Norman, OK 73019, USA[2]

ARTICLE INFO

Article history:
Received 27 February 2015
Revised 9 November 2015
Accepted 25 November 2015
Available online 7 January 2016

Keywords:
Numerical cognition
Estimation
Learning
Cognitive development

ABSTRACT

Children's number-line estimation has produced a lively debate about representational change, supported by apparently incompatible data regarding descriptive adequacy of logarithmic (Opfer, Siegler, & Young, 2011) and cyclic power models (Slusser, Santiago, & Barth, 2013). To test whether methodological differences might explain discrepant findings, we created a fully crossed 2 × 2 design and assigned 96 children to one of four cells. In the design, we crossed anchoring (free, anchored) and sampling (over-, even-), which were candidate factors to explain discrepant findings. In three conditions (free/over-sampling, free/even-sampling, and anchored/over-sampling), the majority of children provided estimates better fit by the logarithmic than cyclic power function. In the last condition (anchored/even-sampling), the reverse was found. Results suggest that logarithmically-compressed numerical estimates do not depend on sampling, that the fit of cyclic power functions to children's estimates is likely an effect of anchors, and that a mixed log/linear model provides a useful model for both free and anchored numerical estimation.

1. Introduction

In this paper, we attempt to reconcile seemingly incompatible data (Barth & Paladino, 2011; Opfer & Siegler, 2007; Opfer, Siegler, & Young, 2011; Slusser, Santiago, & Barth, 2013) regarding the psychophysical functions that link numbers to children's estimates of numerical magnitude.

The psychophysical functions that link numbers to subjects' estimates of numerical magnitude are both theoretically and practically important. Of theoretical interest, functions generating young children's numerical magnitude estimates have been observed in non-symbolic number discrimination of a wide range of species (for review, see Nieder & Dehaene, 2009), to change abruptly with limited experience (Izard & Dehaene, 2008; Opfer & Siegler, 2007), and to closely track abilities to deal with numbers in other contexts (Booth & Siegler, 2006; Thompson & Siegler, 2010). Thus, just as animals can better discriminate 1 and 10 objects than 101 and 110 objects, so too do children estimate magnitudes of symbols 1 and 10 to differ more than 101 and 110. These results suggest that (1) across development, numerical symbols are linked to an innate "mental number line" that allows infants and other animals to discriminate numbers and match them across modalities (see Fig. 1) and (2) linking between symbolic numbers and mental magnitudes is plastic and undergoes significant change (Opfer & Siegler, 2012).

Psychophysical functions linking numbers and estimates of numerical value have also emerged as practically important. Specifically, functions generating children's numerical estimates correlate highly with real-world behavior, including children's memory for numbers, ability to learn arithmetic facts, math grades in school, and math achievement scores (Booth & Siegler, 2006, 2008; Fazio, Bailey, Thompson, & Siegler, 2014; Siegler & Thompson, 2014; Siegler, Thompson, & Schneider, 2011). These findings suggest that children's representations of numerical magnitude play an important role in development of mathematical ability and should be a target for educational interventions.

What psychophysical functions are most likely to generate estimates of numerical value? Across a wide range of tasks and age groups (for review, see Opfer & Siegler, 2012), we have observed two functions as being most likely contenders: the logarithmic function given by Fechner's Law and a standard linear function (see Appendix A)[3]. For example, on number-line estimation tasks,

* Corresponding author.
E-mail addresses: opfer.7@osu.edu (J.E. Opfer), cthomp77@kent.edu (C.A. Thompson).

[1] Permanent address: Kent State University, 228 Kent Hall, Kent, OH 44240, USA.

[2] Address where work completed.

[3] Another model of the logarithmic-to-linear shift, suggested by Anobile, Cicchini, and Burr (2012) and Cicchini, Anobile, and Burr (2014), also combines the two equations into a single formula: $y = a\,((1-\lambda)X + \lambda(U/\ln(U))\ln(X))$, where a is a scaling parameter, λ is the logarithmicity of the estimate, and U is the upper bound of the number-line.

http://dx.doi.org/10.1016/j.cognition.2015.11.015
0010-0277/

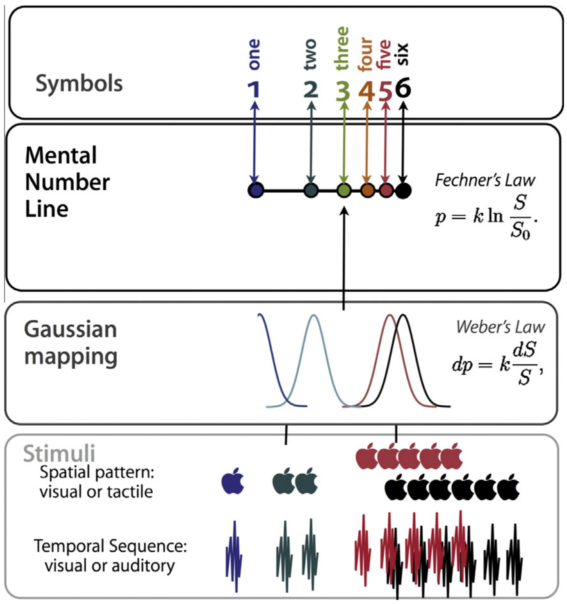

Fig. 1. Model of early numerical magnitude representations (from Opfer & Siegler, 2012).

children are shown a blank line flanked by two numbers (e.g., 0 and 1000) and asked to estimate the position of a third number. Because line length itself is not psychophysically compressive or expansive (Lu & Dosher, 2013), the task provides a relatively straightforward method for assessing compression in numerical magnitude representations.

In many number-line estimation studies, a logarithmic-to-linear shift has been observed. For example, on a 0–1000 task, second graders' median estimates were best fit by a logarithmic function, whereas sixth graders' and adults' median estimates were best fit by the linear function; similarly, over 90% of individual second graders' estimates were better fit by the logarithmic than linear function, whereas the reverse was true of sixth graders and adults (Siegler & Opfer, 2003). This developmental sequence has been observed at different ages with different numerical ranges. It occurs between preschool and kindergarten for the 0–10 range, between kindergarten and second grade for the 0–100 range, between second and fourth grade for the 0–1000 range, and between third and sixth grade for the 0–100,000 range (Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010; Opfer & Siegler, 2007; Siegler & Booth, 2004; Thompson & Opfer, 2010). Similar transitions occur roughly a year later for children with mathematical learning difficulties (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007). Timing of the changes corresponds to periods when children are gaining extensive exposure to numerical ranges: through counting during preschool for numbers up to 10, through addition and subtraction between kindergarten and second grade for numbers through 100, and through all four arithmetic operations in later elementary school.

Against the idea of a logarithmic-to-linear shift, however, Barth and colleagues (Barth & Paladino, 2011; Slusser et al., 2013) have recently presented evidence that estimates of numerical value may follow cyclic power functions rather than being truly Fechnerian logarithmic functions or arithmetically correct linear functions. For example, on a 0–100 number line task, estimates of 7-year-olds were found to follow a 2-cycle power function originally described by Hollands and Dyre (2000). Indeed, fit of the 2-cycle power function was strongest for 7-year-olds' (R^2 = .968) and 8-year-olds' (R^2 = .995) estimates on the 0–1000 number line task, which we examine in our present study. Further, rather than observing an abrupt, single-trial increase in linearity (as reported in Opfer & Siegler, 2007), Barth and colleagues observed a gradual, age-related increase in value of the exponent of the power function. If true, these quantitative findings are theoretically important. First, they suggest that commonalities between estimates of symbolic and non-symbolic magnitude may be illusory, with estimates of symbolic magnitude being affected by children's prior knowledge of proportions (e.g., 500 is half of 1000). Second, they suggest that changes in numerical magnitude estimates are *quantitative* (in the sense that one parameter in the same function changes over time) rather than *qualitative* (in the sense that different functions are needed to describe younger versus older children's estimates).

1.1. Why different functions? Sampling versus anchoring

To illustrate differences between data cited in support of the logarithmic-to-linear shift account and the proportion-judgment account, it is useful to compare 7- and 8-year-olds' number-line estimates on the 0–1000 task (Fig. 2), where Slusser et al. (2013) found a better fit for the 2-cycle power function over the logarithmic, despite the logarithmic function providing a better fit in data collected by Opfer and Siegler (2007). Given that children's ages and numeric ranges were the same, something must explain these discrepant findings.

One potential cause of the discrepancy is methodological differences in *sampling* (Barth, Slusser, Cohen, & Paladino, 2011; Slusser et al., 2013), with fit of the logarithmic function being an artifact of sparsely sampling at the upper ranges (e.g., obtaining few estimates for numbers 750–1000) and heavily sampling at lower ranges (e.g., obtaining many estimates for numbers 0–250). As Slusser et al. (2013) write, "there is a resounding tendency for researchers to sample heavily from the lower end of the number line and scarcely from the upper end.... This practice focuses on participants' propensity to overestimate small numbers, but yields little data to reveal the details of underestimation patterns for larger numbers" (p. 4). This observation has potential force. As can be seen in Fig. 2, Opfer and Siegler (2007) collected estimates for 13 numbers in the 0–250 range and 3 numbers in the 750–1000 range, whereas Slusser et al. (2013) collected estimates for 7 numbers in each range.

Another potential cause of the discrepancy is methodological differences in *anchoring* (Opfer et al., 2011), with fit of the 2-cycle power function being an effect of experimenters telling children the placement of 500. In the typical number-line task (Siegler & Booth, 2004; Siegler & Opfer, 2003, Exp. 1; Booth & Siegler, 2006, Exp. 2; Laski & Siegler, 2007; Opfer & Martens, 2012; Opfer & Siegler, 2007; Opfer & Thompson, 2008; Thompson & Opfer, 2008; though see Siegler & Booth, 2004, Exp. 2, and Booth & Siegler, 2006, Exp. 1, for use of anchors), children are given no supervision on any of their number line placements. In contrast, in all studies finding a superior fit of the 2-cycle power function, children's estimate of the halfway point is anchored. For example, in Slusser et al. (2013), children were told, "Because 500 is half of 1000, it goes right in the middle between 0 and 1000. So 500 goes right there, but it's the only number that goes right there." Given previous training studies (Opfer & Siegler, 2007; Opfer & Thompson, 2008; Thompson & Opfer, 2008) finding that a single trial of feedback can increase linearity of estimates, such anchors seem highly likely to affect children's estimates.

While these two potential causes of the discrepancy in findings are not mutually exclusive, each cause has different theoretical implications. From the logarithmic-to-linear-shift account, differences in sampling are predicted to be minor because oversampling has only a small impact on absolute fits and no impact on model selection. In contrast, from the proportion-judgment account, dif-