Accepted Manuscript

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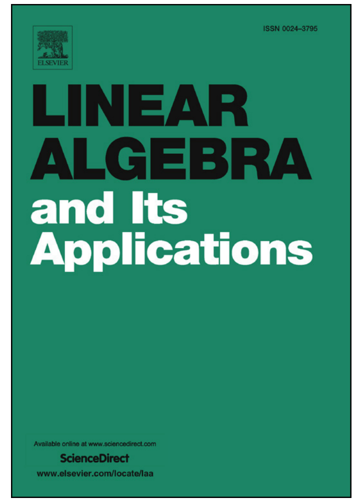

PII: S0024-3795(18)30432-4
DOI: https://doi.org/10.1016/j.laa.2018.09.005
Reference: LAA 14720

To appear in: *Linear Algebra and its Applications*

Received date: 25 April 2018
Accepted date: 4 September 2018

Please cite this article in press as: A. Chistopolskaya, On nilpotent generators of the Lie algebra $\mathfrak{sl}_n$, *Linear Algebra Appl.* (2018), https://doi.org/10.1016/j.laa.2018.09.005

ON NILPOTENT GENERATORS OF THE LIE ALGEBRA $\mathfrak{sl}_n$

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Abstract. Consider the special linear Lie algebra $\mathfrak{sl}_n(\mathbb{K})$ over an infinite field of characteristic different from 2. We prove that for any nonzero nilpotent X there exists a nilpotent Y such that the matrices X and Y generate the Lie algebra $\mathfrak{sl}_n(\mathbb{K})$.

1. Introduction

It is an important problem to find a minimal generating set of a given algebra. This problem was studied actively for semisimple Lie algebras. In 1951, Kuranishi [5] observed that any semisimple Lie algebra over a field of characteristic zero can be generated by two elements. Twenty-five years later, Ionescu [4] proved that for any nonzero element X of a complex or real simple Lie algebra $\mathcal{G}$ there exists an element Y such that the elements X and Y generate the Lie algebra $\mathcal{G}$. In the same year, Smith [6] proved that every traceless matrix of order $n \geqslant 3$ is the commutator of two nilpotent matrices. These results imply that the special linear Lie algebra $\mathfrak{sl}_n$ can be generated by three nilpotent matrices. In 2009, Bois [2] proved that any simple Lie algebra in characteristic different from 2 and 3 can be generated by two elements. He also extended Ionescu's result for classical simple Lie algebras over field of characteristic different from 2 and 3 and proved that graded Cartan type Lie algebra $W(m, \underline{\boldsymbol{n}})$ has this property if and only if $m = 1$. In 2010, Bois [3] showed that for simple graded Lie algebras of Cartan type S, H or K Ionescu's result does not hold.

In this paper we obtain an analogue of Ionescu's result for nilpotent generators of the Lie algebra $\mathfrak{sl}_n$.

Theorem 1. *Let $\mathbb{K}$ be an infinite field of characteristic different from 2. For any nonzero nilpotent X there exists a nilpotent Y such that the matrices X and Y generate the Lie algebra $\mathfrak{sl}_n(\mathbb{K})$.*

Our interest to this subject was motivated by the study of additive group actions on affine spaces, see [1, Theorem 5.17]. In a forthcoming publication we plan to extend Theorem 1 to arbitrary simple Lie algebras.

The author is grateful to her supervisor Ivan Arzhantsev for posing the problem and permanent support. Also she would like to thank the referee for valuable suggestions.

2. Main results

Let $\mathbb{K}$ be an infinite field with $\operatorname{char} \mathbb{K} \neq 2$. A set of elements $\lambda_1, \ldots, \lambda_n$ ($\lambda_i \in \mathbb{K}$) is called *consistent* if the following conditions hold:

(1) $\lambda_1 + \ldots + \lambda_n = 0$;
(2) $\lambda_i \neq 0$ for all i;
(3) $\lambda_i \neq \lambda_j$ for all $i \neq j$;

2010 *Mathematics Subject Classification.* Primary 17B05, 17B22; Secondary 15A04.
Key words and phrases. Nilpotent matrix, commutator, Lie algebra, generators.