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A coloring property for stable allocations

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HIGHLIGHTS

- The anti-blocking property for stable allocations and stable flows is proved.
- This is an improvement of previously known results for stable b-matchings.
- The results are also extended to stable flows and allocations with choice functions.

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ABSTRACT

We prove that in the stable allocations problem for a fixed vertex v there can be done a partition of the edges incident with v such that in any stable allocation there is at most one edge incident with v from each class. This is an improvement of the coloring theorem for stable b-matchings given in Fleiner (2003). We also extend our result to stable flows and allocations with choice functions.

1. Introduction

In 1962 David Gale and Lloyd Shapley proved in Gale and Shapley (1962), a relatively short paper with only 7 pages, a theorem which later turned out to have far-reaching implications in many different fields, and because of which Shapley was eventually awarded the Nobel Prize in Economics in 2012.

They formally stated and solved the following question called *the stable marriage problem* or *the stable matching problem*. Given n men and n women such that each person has a strict preference list of the members of the opposite gender, is there a way to marry everybody such that there exists no pair (m, w) which could destabilize the marriages, that is, a man m and a woman w who prefer each other to their current partners.

In the last decades, the stable matching problem has evolved to a theory studying many different settings as well as many different concepts of stability. The paper of Gale and Shapley not only started a new theory but the algorithm they gave is being used again and again to solve more complex matching problems. We will here introduce only some of them, the ones that are relevant for this paper. For a detailed and encompassing introduction to the matching theory we refer the reader to Gusfield and Irving (1989).

As the title of Gale and Shapley (1962) indicates, the original problem that Gale and Shapley attacked was the *college admissions problem* which is different from the stable matching problem in that a college can admit more that one student. This problem is also called *the many-to-one matching problem* and is actually a straightforward generalization of the stable matching problem. The number of students that a college can admit is called *the quota* of that college.

If we allow the vertices on both sides of the bipartite graph to have quotas ≥ 1 then we get *the many-to-many matching problem* or the *b-matching problem*, where b stands for the quota function which assigns a natural number to every vertex. For instance, if we allow students to study at multiple universities simultaneously, the college admission problem becomes a (proper) many-to-many problem. All of the mentioned problems have been solved with positive answers. There actually exist algorithms which list all the stable solutions of a given problem, no matter what the preference lists are.

The next milestone in the theory of stable matchings is *the stable allocation problem* of Baïou and Balinski defined in Baïou and Balinski (2002). In this problem we again have a bipartite graph and preference lists of the vertices over the vertices on the other side. This time we allow the quotas to be positive real numbers. The edges are also assigned upper bounds called *capacities*. An *allocation* is a function which assigns every edge a positive value such that quotas and capacities are respected. This problem is often illustrated by firms and workers where the allocation corresponds

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to the number of hours a worker is employed in a firm. The concept of stability together with the other formal definitions will be given in the next section. Note that the original stable matching problem is a particular case of the stable allocation problem in which all edges have the capacity 1 and all quotas are 1.

Finally, Fleiner introduced in Fleiner (2014) *the stable flow problem* where the bipartite graph is substituted by a network and the edges are oriented arcs. The stable flow problem generalizes the stable allocation problem. The formal definition will be given in Section 5.

As we have seen, the above generalizations have been obtained by introducing quotas >1 and by letting them take values in $\mathbb{R}$, as well as by passing from bipartite graphs to arbitrary digraphs. Generalizations, however, have been achieved in two different directions simultaneously. The other direction is based on yet another breakthrough in the theory of stable matchings, the one that introduced the so-called *choice functions* which allow a vertex to have much more complex preferences rather than a linear preference list over the edges incident with it.

In 2003 Tamás Fleiner proved in Fleiner (2003) the following coloring property for b-matchings: for a fixed vertex z with quota q, the edges incident with z can be partitioned into q many classes such that in no stable b-matching there are 2 edges from the same class. Fleiner asked if there can be found a similar coloring property for allocations. In Section 4 we give a positive answer to his question.

Moreover, we also prove two generalizations of our result. The first one is for stable flows and is based on Fleiner's method from Fleiner (2014) for translating any stable flow problem into a stable allocation problem. The second one concerns stable allocations with choice functions. This much more general result is proven partially, assuming an extra hypothesis.

The theorem in Fleiner (2003) is based on *the comparability theorem* of Roth and Sotomayor proved in Roth and Sotomayor (1989). In Section 3 we show that surprisingly easily a similar theorem can be proven for allocations as well.

2. Preliminaries

Baïou and Balinski defined in Baïou and Balinski (2002) the stable allocation problem (the notation in this definition is taken from Fleiner, 2014):

Let $G = (W \cup F, E)$ be a bipartite graph. W and F are referred to as sets of workers and firms. A map $q : W \cup F \to \mathbb{R}$ determines the *quotas* of the vertices. A map $p : E \to \mathbb{R}$ determines the *capacities* of the edges in E. For each $v \in W \cup F$ we are given a preference list $<_v$ of v over the edges in E incident with v ($e' <_v e$ means that v prefers e' to e).

An *allocation* is a nonnegative map $g : E \to \mathbb{R}$ such that $g(e) \leq p(e)$ for all $e \in E$, and for any $v \in W \cup F$ we have $\sum_{vx \in E} g(vx) \leq q(v)$. If the last is an equality, we say that v is *g-saturated*.

An allocation is said to be *stable* if every edge $wf \in E$ satisfies one of the following:

1. $g(wf) = p(wf)$ (full capacity employment),
2. worker w is g-saturated and w does not prefer f to any of his employers,
3. firm f is g-saturated and f does not prefer w to any of its employees.

If g_1 and g_2 are allocations and $w \in W$, we say that g_1 *dominates g_2 for worker w*, denoted $g_1 \preceq_w g_2$, if one of the following holds:

1. $g_1(wf) = g_2(wf)$ for every $f \in F$,
2. (a) $\sum_{f' \in F} g_1(wf') = \sum_{f' \in F} g_2(wf') = q(w)$, and
   (b) $g_1(wf) < g_2(wf)$ and $g_1(wf') > 0$ implies $wf' \leq_w wf$.

This means that either g_1 and g_2 are the same for w or w is both g_1-saturated and g_2-saturated and if g_1 has less allocation than g_2 on some edge then that edge is at most w's least preferred positively allocated edge in g_1. A similar definition can be given for $f \in F$.

Besides proving the existence of a stable allocation for any given W, F, E, p, q, and showing that stable allocations have a natural lattice structure, Baïou and Balinski proved in Baïou and Balinski (2002) that if g_1 and g_2 are two stable allocations and $v \in W \cup F$, then $g_1 \preceq_v g_2$ or $g_2 \preceq_v g_1$. We will refer to this property as the *local linear ordering* of stable allocations. It will be the decisive tool in the rest of the paper.

3. The comparability theorem for allocations

Roth and Sotomayor proved in Roth and Sotomayor (1989) the following Comparability Theorem:

Theorem 1. *Let M and M' be two stable b-matchings for a bipartite preference system. Let z be a vertex and $M_z := M \cap D(z)$, $M'_z := M' \cap D(z)$, where $D(z)$ is the set of edges incident with z. If $M_z \neq M'_z$, then $|M_z| = |M'_z| =: b(z)$ and the $b(z)$ $<_z$-best edges of $M_z \cup M'_z$ are either M_z or M'_z.*

In their setting the b-matchings are a special case of allocations, where the capacities are all equal to 1 and the quotas are given by a function $b : W \cup F \to \mathbb{N}$. The b-matchings are also called many-to-many matchings.

Using the existence of the local linear ordering for a fixed vertex v, we immediately get a similar result in the broader context of stable allocations:

Theorem 2. *Let g_1 and g_2 be two stable allocations. Let $v \in W \cup F$. Let $M_{g_1} := \{e \in E | g_1(e) > 0\}$ and $M_{g_2} := \{e \in E | g_2(e) > 0\}$ be the supports of the allocations g_1 and g_2. Let $M_1 := M_{g_1} \cap D(v)$, $M_2 := M_{g_2} \cap D(v)$, where $D(v)$ is the set of edges incident with v. If $g_1 \preceq_v g_2$, then the $|M_1|$ $<_v$-best edges of $M_1 \cup M_2$ are M_1.*

Proof. Without loss of generality suppose that $v \in W$. It suffices to show that there do not exist edges $vf <_v vf'$ such that $vf \in M_2 \backslash M_1$ and $vf' \in M_1$. If there existed such edges, this would mean that $g_1(vf) = 0 < g_2(vf)$, and $g_1(vf') > 0$. But this would be in contradiction with $g_1 \preceq_v g_2$, because g_1 can be less than g_2 at most on v's least preferred positively allocated edge in g_1, and that is not the case for vf, since $g_1(vf') > 0$.

In the context of stable allocations we cannot have the '$|M_z| = |M'_z| = b(z)$' part of the Roth–Sotomayor theorem, since the cardinalities $|M_1|$ and $|M_2|$ might be different for different allocations g_1 and g_2. However, we know from Baïou and Balinski (2002) that the sum of allocations at a fixed vertex, say $w \in W$, is the same in all stable allocations, that is, $g(w) := \sum_{wf \in E} g(wf)$ is a constant.

4. The coloring property

Fleiner observed in Fleiner (2003) that the Roth–Sotomayor theorem has the following nice corollary for stable b-matchings:

Theorem 3. *Given a bipartite preference system together with a quota function $b : W \cup F \to \mathbb{N}$, for any vertex z there is a partition of $D(z)$ into $b(z)$ parts $D_1(z), D_2(z), \ldots, D_{b(z)}(z)$ so that $|M \cap D_i(z)| \leq 1$ for any stable b-matching M and any $i \in \mathbb{N}$ with $1 \leq i \leq b(z)$.*

A similar coloring can be given for allocations and proven directly using the local linear ordering of some fixed vertex v: