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On the reversal bias of the Minimax social choice correspondence☆

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HIGHLIGHTS

- Three versions of the reversal bias for social choice correspondences are proposed.
- A new description of Minimax correspondence is proposed.
- We characterize when Minimax correspondence suffers the reversal bias of each type.
- Condorcet and Borda correspondences are immune to the reversal bias of each type.
- Graph theory is used as the main tool.

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ABSTRACT

We introduce three different qualifications of the reversal bias in the framework of social choice correspondences. For each of them, we prove that the Minimax social choice correspondence is immune to it if and only if the number of voters and the number of alternatives satisfy suitable arithmetical conditions. We prove those facts thanks to a new characterization of the Minimax social choice correspondence and using a graph theoretical approach. We discuss the same issue for the Borda and Copeland social choice correspondences.

1. Introduction

Consider a committee having $h \geq 2$ members who have to select one or more elements within a set of $n \geq 2$ alternatives. Usually, the procedure used to make that choice only depends on committee members' preferences on alternatives. We assume that preferences of committee members are expressed as strict rankings (linear orders) on the set of alternatives, and call preference profile any list of h preferences, each of them associated with one of the individuals in the committee. Thus, a procedure to choose, whatever individual preferences are, one or more alternatives as social outcome can be represented by a social choice correspondence (scc), that is, a function from the set of preference profiles to the set of nonempty subsets of the set of alternatives.

The assessment of different sccs and their comparison is usually based on which properties, among the ones considered desirable or undesirable under a social choice viewpoint, those sccs fulfil. Moving from the ideas originally proposed by Saari (1994) and then deepened by Saari and Barney (2003), we focus here on a quite unpleasant property that a scc may meet and that, in our opinion, has not deserved the right attention yet.

In order to describe such a property, recall that the reversal of a preference profile is the preference profile obtained by it assuming a complete change in each committee member's mind about her own ranking of alternatives (that is, the best alternative gets the worst, the second best alternative gets the second worst, and so on). Assume now that a given scc associates with a certain preference profile a singleton, that is, it selects a unique alternative. If we next consider the outcome determined by the reversal of the considered preference profile, we would expect to have something different from the previous singleton as it seems natural to demand a certain degree of difference between the outcomes associated with a preference profile and its reversal. As suggested by Saari and Barney (2003, p.17),

> suppose after the winner of an important departmental election was announced, it was discovered that everyone misunderstood the chair's instructions. When ranking the three candidates, everyone listed his top, middle, and bottom-ranked candidate in the natural order first, second, and third. For reasons only the chair understood, he expected the voters to vote in

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the opposite way. As such, when tallying the ballots, he treated a first and last listed candidate, respectively, as the voter's last and first choice. Imagine the outcry if after retallying the ballots the chair reported that [...] the same person won.

In other words, common sense suggests that we should express doubts about the quality of a scc which associates the same singleton both with a preference profile and with its reversal, that is, which suffers what we are going to call the reversal bias.

Among the classical sccs , such a bias is experienced by the Minimax scc, also known as Simpson–Kramer or Condorcet scc, that is, the scc which selects those alternatives whose greatest pairwise defeat is minimum. Indeed, assume that a committee having six members ($h = 6$) has to select some alternatives within a set of four alternatives denoted by 1, 2, 3 and 4 ($n = 4$). Consider then a preference profile represented by the matrix

$$\begin{bmatrix} 4 & 4 & 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 & 3 & 1 \\ 2 & 3 & 1 & 3 & 1 & 2 \\ 3 & 1 & 2 & 4 & 4 & 4 \end{bmatrix}$$

where, for every $i \in \{1, 2, 3, 4, 5, 6\}$, the ith column represents the ith member's preferences according to the rule that the higher the alternative is, the better it is. A simple check shows that the Minimax scc associates both with that preference profile and with its reversal the same set $\{4\}$. On the other hand, if we consider two alternatives only, then the Minimax scc agrees with the simple majority and it is immediate to verify that it is immune to the reversal bias whatever the number of committee members is.

For such a reason, we address the problem of finding conditions on the number of individuals and on the number of alternatives that make the Minimax scc immune to the reversal bias. Our main result[2] is the following theorem.

Theorem A. *The Minimax* scc *is immune to the reversal bias if and only if* $h \leq 3$ *or* $n \leq 3$ *or* $(h, n) \in \{(4, 4), (5, 4), (7, 4), (5, 5)\}$.

Theorem A shows, in particular, that the Minimax scc does not exhibit the reversal bias not only when there are two alternatives but also in other cases. Remarkably, that property holds true when alternatives are three, independently on the number of individuals, and when individuals are three, independently on the number of alternatives.

The proof of Theorem A requires a certain amount of work and the use of language and methods taken from graph theory.[3] Indeed, standard social choice theoretical arguments naturally allow one to prove that, for lots of pairs (h, n), the Minimax scc suffers the reversal bias.[4] On the other hand, except for the trivial case $n = 2$, they turn out to be difficult to apply to prove that, for the remaining pairs, the Minimax scc is immune to the reversal bias. In particular, no simple intuition indicates how to treat the cases $(h, n) \in \{(4, 4), (5, 4), (7, 4), (5, 5)\}$. For such a reason, we first propose a new characterization of the Minimax scc showing that, for every preference profile, an alternative x is selected by the Minimax scc if and only if, for every majority threshold μ not exceeding the number of individuals but exceeding half of it, if there is an alternative which is preferred by at least μ individuals to x, then, for every alternative, there is another one which is preferred by at least μ individuals to it (Proposition 1). We then associate with each preference profile p and each majority threshold μ a directed graph $\Gamma_\mu(p)$, called a majority graph, whose vertices are the alternatives and whose arcs are the μ-majority relations among alternatives (Section 5.2). By the analysis of connection and acyclicity properties of those graphs, we find out a general and unified method to approach the proof of Theorem A. That allows, in particular, to avoid the repetition of similar arguments and the discussion of very long lists of cases and subcases. The geometric representation of the graph $\Gamma_\mu(p)$ is also a useful mental guidance in the tricky steps needed to carry on such an analysis as well as the proof of Theorem A. We emphasize that the results related to graph theory deal with quite general majority issues so that they are not limited, in their meaning, to the specific problem considered in the paper. We are confident that those results could be a smart tool to manage, in the future, many other problems.

We also introduce two weaker versions of reversal bias. Namely, we say that a scc suffers the reversal bias of type 2 if there exists a preference profile such that the outcomes associated with it and its reversal are not disjoint and one of the two is a singleton; we say instead that a scc suffers the reversal bias of type 3 if there exists a preference profile such that the outcomes associated with it and its reversal are not disjoint and none of the two is the whole set of the alternatives. It is immediate to observe that the reversal bias (also called reversal bias of type 1) implies the reversal bias of type 2 which in turn implies the reversal bias of type 3. Using the same tools and techniques used to prove Theorem A, we get the following results.[5]

Theorem B. *The Minimax* scc *is immune to the reversal bias of type* 2 *if and only if* $h = 2$ *or* $n \leq 3$ *or* $(h, n) = (4, 4)$.

Theorem C. *The Minimax* scc *is immune to the reversal bias of type* 3 *if and only if* $n = 2$ *or* $(h, n) = (3, 3)$.

We emphasize that there is an interesting link between the different qualifications of reversal bias above described and the concept of Condorcet loser. Indeed, let C be a scc satisfying the Condorcet principle, that is, always selecting the Condorcet winner as unique outcome when it exists. If C is immune to the reversal bias of type 1, then it never selects the Condorcet loser as the unique outcome, that is, C fulfils the weak Condorcet loser property; if C is immune to the reversal bias of type 2, then it never selects the Condorcet loser, that is, C fulfils the Condorcet loser property. Thus, since the Minimax scc satisfies the Condorcet principle, Theorems A and B provide, in particular, conditions on (h, n) that are sufficient to make the Minimax scc satisfy the weak Condorcet loser property and the Condorcet loser property, respectively. Certainly, as it is not known whether such conditions are also necessary, determining all the pairs (h, n) making the Minimax scc satisfy those properties is an interesting problem which, in our opinion, can be fruitfully attacked using the methods described in this paper. Finally note that, given a scc C always selecting the Condorcet winner (not necessarily as the unique outcome) when it exists, we have that if C is immune to the reversal bias of type 2, then it fulfils the weak Condorcet loser property; if C is immune to the reversal bias of type 3, then it never selects the Condorcet loser when the set of outcomes is different from the whole set of alternatives.

Observe now that, even though the main concepts of our paper are mainly inspired by the ideas of Saari and Barney (2003), the framework we consider, as well as the terminology we use, is different from the one they used. Indeed, they deal with election methods, namely, functions from the set of finite sequences of individual preferences (still called preference profiles) to the set of complete and transitive relations on the set of alternatives. In that

[2] Theorem A is a rephrase of Theorem 2 for $j = 1$.

[3] Note that the use of graphs in social choice theory is well established (see, for instance, Laslier, 1997).

[4] See Propositions 23 and 24 and related comments.

[5] Theorems B and C are rephrases of Theorem 2 for $j = 2$ and $j = 3$, respectively.