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Comparing preference orders: Asymptotic independence☆

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HIGHLIGHTS

- A decision maker faces choice between two preference orders over n objects.
- He adopts a "comparison rule" that specifies how to compare preference orders.
- We examine statistical correlation between choices by different comparison rules.
- Several comparison rules induce almost independent choices when n is large.

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ABSTRACT

A decision maker is presented with two preference orders over n objects and chooses the one which is "closer" to his own preference order. We consider several plausible comparison rules that the decision maker might employ. We show that when n is large and the pair of orders to be compared randomly realizes, different comparison rules lead to statistically almost independent choices. Thus, two people with a common preference relation may nonetheless exhibit almost uncorrelated choice patterns.

1. Introduction

Consider a questionnaire that asks a voter: "Which candidate, A or B, has a policy preference closer to yours?" Unless the meaning of "closer" is provided in advance, even voters with the same policy preference may respond differently. Yet, if the above question is repeated for a sufficiently large variety of candidates and two voters exhibit significantly distinct choice patterns, one might be tempted to conclude that they have distinct policy preferences.

This paper shows a case where in a general situation like the above, almost uncorrelated choice patterns arise from individuals with a common underlying preference. We consider several *comparison rules* that map the decision maker's underlying preference over objects (e.g., the voter's preference over policies) to his ranking over preference orders (e.g., the voter's ranking over candidates). One is the *Kemeny rule* which is based on the *Kemeny distance* of each order from the decision maker's own order (Kemeny and Snell, 1962), and the other are *lexicographic rules* which are procedurally simpler. These comparison rules share certain axioms and, indeed, frequently induce identical comparisons when the number n of objects is small. We show that for some rules this similarity disappears as n becomes large. More precisely, when the pair of preference orders to be compared randomly realizes, for large n, the comparisons made under some rules are almost statistically (pairwise) independent. There are of course comparison rules that exhibit positive correlation bounded away from zero. We give an example of such rules in an Appendix to the paper.

One way to interpret this result is to assume that the decision maker has some "welfare" preference, as distinguished from his "behavioral" preference, both defined on the set of preference orders (see Rubinstein and Salant, 2012). For instance, suppose a voter must cast a ballot for either candidate A or B. Suppose further that after the election, the winning candidate will face the choice between any two policies equally likely. Hence the voter's welfare preference over candidates would be the one which is defined by the Kemeny rule applied to his policy preference. However, the actual choices made by the voter may be inconsistent with this preference. For example, he may adopt the quicker decision procedure in which he first compares the top policies for the two candidates according to his own policy preference. If he prefers A's top policy then he votes for A; if he is indifferent, then he compares the second-ranked policies for the candidates, and so on. This procedure induces choices that are consistent

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with a lexicographic rule. Our result suggests that when the number of policies is large, the distortion in voting behavior caused by employing the alternative procedure is quite large: the behavioral preference coincides with the welfare preference only with probability around one-half. But in the Appendix, we also indicate a different lexicographic procedure that approximates the Kemeny rule with much higher accuracy.

Some of the comparison rules considered in this paper have already appeared in previous studies, especially in the literature on the incentive problem arising in preference aggregation. This literature addresses the question of whether and how society can prevent individuals from manipulating the social preference. To describe this problem, one needs to make an assumption about how each individual ranks possible social preference orders based on his own preference order. Bossert and Storcken (1992) assume that individuals follow the Kemeny rule. Sato (2013c) allows for a wide class of comparison rules which include all rules considered here. Bossert and Sprumont (2014) assume a "domination"-type rule. In the case of a voter's comparison between candidates A and B, we say that A *dominates* B (in terms of the voter's policy preference) if, whenever the voter agrees with candidate B on a pair of policies, he also agrees with candidate A on that pair of policies. A comparison rule is said to satisfy the *domination axiom* if, whenever candidate A dominates B, the voter prefers A to B. The comparison rules discussed in this paper are complete and satisfies the domination axiom; hence they are different complete extensions of the domination rule.

In a somewhat different framework from this paper's, Laffond and Lainé (2000) characterize two lexicographic comparison rules by three axioms: *Neutrality*, *Independence*, and *Tournament Consistency*. The first two axioms seem more or less standard in the literature. Tournament Consistency requires consistency between the majority preference over objects and the majority preference over orders, when all individuals employ the same comparison rule.[1] Despite such qualitative similarity, we show that the two lexicographic rules are asymptotically independent.

In a related vein, Lainé et al. (2014) introduce a condition on a social welfare function called *hyper-stability*. Roughly speaking, this condition requires that if the social welfare function aggregates individuals' preferences over objects into a social preference order over objects, then the function aggregates individuals' preferences over orders (over objects) into that social preference order, where individuals are assumed to follow some comparison rule satisfying the domination axiom mentioned above. Lainé et al. also provide a survey of the literature on analysis of preferences over orders.

The rest of the paper is organized as follows. In Section 2, we formally define comparison rules. In Section 3, we show that five rules are asymptotically independent. In Section 4, we contrast the result obtained for large n with the case of small n, and conclude the paper. An Appendix to the paper provides an example of a rule that is asymptotically correlated with the Kemeny rule.

2. Comparison rules

The following voting example illustrates our model.

Example. Suppose there are three political goals, {Defense, Equality, Growth}. A voter has the priority order: Equality $\succ$ Growth $\succ$ Defense. Two political candidates, A and B, have the priority orders, Defense $\succ_A$ Equality $\succ_A$ Growth, and Growth $\succ_B$ Defense $\succ_B$ Equality, respectively. Which candidate does the voter choose? The following are examples of a *comparison rule* that the voter might employ.

Kemeny rule. The voter measures his *Kemeny distance* from each candidate (i.e., the number of pairs of goals on which he disagrees with the candidate) and chooses the candidate closer to him. His distance from each candidate is 2, since he disagrees with A on {Defense, Equality} and {Defense, Growth}, and disagrees with candidate B on {Equality, Growth} and {Defense, Equality}. Thus in this case, the rule does not select a single candidate.

Descending rule.[2] The voter first compares the top-priority goals for candidates A and B (i.e., Defense and Growth) according to his own priority order. If he ranks one candidate's top goal higher then he chooses that candidate; if he is indifferent then he goes on to compare the second-priority goals for the two candidates, and so on. As Growth $\succ$ Defense, the voter chooses candidate B.

Ascending rule. The voter first compares the *bottom*-priority goals for candidates A and B (i.e., Equality and Growth). If he ranks one candidate's bottom goal *lower* then he votes for that candidate; if he is indifferent then he goes on to compare the second-priority goals for the two candidates, and so on. As Equality $\succ$ Growth, the voter chooses candidate A.

Inverse descending rule. The voter first compares the *ranks* which his top-priority goal (Equality) receives from the two candidates. If one candidate ranks it higher then he chooses that candidate; if the two candidates rank it equally then he goes on to compare the ranks which his second-priority goal receives, and so on. As candidate A ranks Equality higher than B does, the voter chooses A. The *inverse ascending rule* is similarly defined. □

Formally, let $\{1, 2, \ldots, n\}$ be the set of *objects*. In this paper a *(preference or priority) order* refers to a linear order over objects. We fix the decision maker's preference order over objects as

$$1 \succ 2 \succ \ldots \succ n.$$

Any order is expressed as a permutation of $\{1, \ldots, n\}$,

$$\pi = (\pi(1), \ldots, \pi(n)) \quad \text{or simply} \quad \pi = \pi(1)\ldots\pi(n).$$

This notation means that "$\pi(i)$ is the ith-ranked object in the order π". Let Π be the set of orders. In principle, a *comparison rule* is a rule that maps the decision maker's underlying order $\succ$ to his ordering over orders. However, since we have fixed $\succ$ as above, we can identify a comparison rule as an ordering on Π.

An *inversion* in an order π is a pair $\{i, j\}$ of ranks such that $i < j$ and $\pi(i) > \pi(j)$. For each rank i, let $L_i(\pi)$ denote the number of inversions $\{i, j\}$ of the form $i < j$ and $\pi(i) > \pi(j)$. The vector $L(\pi) = (L_i(\pi))_{i=1}^n$ is called the *Lehmer code* of π. It is known that the mapping L is a bijection from Π to $\{0, \ldots, n-1\} \times \{0, \ldots, n-2\} \times \cdots \times \{0\}$. Let $I(\pi)$ be the total number of inversions in π,

$$I(\pi) = \sum_{i=1}^{n} L_i(\pi).$$

The number $I(\pi)$ is called the *Kemeny distance* of the order π from the decision maker's order $12\ldots n$.[3]

We consider the following comparison rules:

- *Kemeny rule* $\succsim_K$: $\pi \succsim_K \pi'$ if and only if $I(\pi) \le I(\pi')$.

[1] See Remark in Section 2 for details on these axioms.

[2] Sato (2013c) discusses the concept of non-manipulability of social preferences when agents follow the descending rule.

[3] See Kemeny and Snell (1962) for an axiomatic characterization of the Kemeny distance function, and Can and Storcken (2013) or Farnoud et al. (2012) for an improved characterization result. Can (2014) also provides an axiomatic characterization of a class of distance functions on orders which have weights assigned to positions in an order.