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Unstable network fragmentation in co-evolution of Potts spins and system topology

Joanna Toruniewska [a], Krzysztof Suchecki [a], Janusz A. Hołyst [a,b,*]

[a] *Center of Excellence for Complex Systems Research, Faculty of Physics, Warsaw University of Technology, Koszykowa 75, PL-00662 Warsaw, Poland*

[b] *ITMO University, 19 Kronverkskiy av., 197101 Saint Petersburg, Russia*

HIGHLIGHTS

- A fragmented network is unstable in quasi-ergodic coevolution.
- The system slowly evolves towards a single fully ordered component.
- Dynamics of clusters "surfaces" is responsible for the transition shape.

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ABSTRACT

We investigate co-evolution of discrete q-state Potts model and the underlying network topology, where spin changes and link re-wiring follow the same canonical ensemble dynamics in a constant temperature. It means that there are no absorbing, frozen states present in our model. Depending on the temperature T and probability of link dynamics p the system can exist in one of three states: ordered, disordered and ordered clusters (fragmented network), with the last being unstable and slowly relaxing into ordered state. The transition from ordered clusters to globally ordered system is characterized by non-exponential, slow growth of the order parameter. We investigate this process analytically and explain the transition characteristics as the result of the dominance of activity of "surface" nodes in each ordered cluster, as opposed to "bulk" nodes that are inactive.

1. Introduction

Many different systems that are subject of research in natural and social sciences can be described as networks [1]. Processes found in such systems can be often described in terms of dynamics on the network, where internal variables of the nodes change, or dynamics of the network, where connections are changing. While most dynamical processes involving networks can be classified as either of those two, an interesting topic of research is the interaction between them, often called co-evolution or adaptive networks [2,3]. Special attention has been given to co-evolution of models displaying emergence of cooperation or same-state domains, such as voter model [4,5], Ising-like model [6], models based on game theory [3] and also oscillator synchronization [7]. An interesting process observed is the *fragmentation transition*, where the system

* Corresponding author at: Center of Excellence for Complex Systems Research, Faculty of Physics, Warsaw University of Technology, Koszykowa 75, PL-00662 Warsaw, Poland.

E-mail address: jholyst@if.pw.edu.pl (J.A. Hołyst).

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splits into topologically separate domains or clusters, each displaying internal order. A common feature found in most of these models is the fact that they have an *absorbing state* and the fragmented network is one of such states. The process depends not only on model parameters, but also on an initial system state [8], since existence of absorbing states means non-ergodicity.

In this work, we investigate the co-evolution of discrete q-state Potts model and rewiring of connection. The Potts model, as a generalization of Ising model, has been widely studied [9]. Of special interest are phase transitions, that have been investigated in various topologies, starting from periodic lattices [9], Bethe lattice [10], Apollonian networks [11] and complex networks [12–14]. Depending on the number of states q, the discrete Potts model is equivalent to the Ising model (for $q = 2$, we discuss this more thoroughly in Section 2), possesses the same properties as percolation ($q \to 1$) [15] or features discontinuous phase transition along the typical continuous one ($q > 2$) [12,14]. The propensity of the Potts model to generate homogeneous domains in low temperatures allowed its application for detecting community structure of a network [16–18]. The inverse Potts model has been also recently applied to reconstruct a real social network (Italian parliament) [19].

In our work, both Potts model and the connection dynamics follow the canonical ensemble under constant temperature. We show with numerical simulations, that unlike in similar models where non-homogeneous absorbing state is present [6], the state of ordered clusters that occurs due to fragmentation is a transient state in finite systems and that the system relaxes into a single ordered cluster. We investigate and explain evolution of the order parameter during this transition.

Section 2 describes the model and its implementation in simulations, Section 3 discusses phase diagram, Section 4 describes all assumptions used, Section 5 shows the analytical description of the phenomena and Section 6 offers summary and concluding remarks.

2. Model

We analyze discrete Potts model on a complex network, where node states are co-evolving with network connections. We consider a fixed set of N nodes, each node i possessing a Potts spin $s_i \in 1, 2, \ldots, q$ with q being model parameter. The nodes are connected by edges, which can be described by an adjacency matrix A, where $A_{ij} = A_{ji} = 1$ when nodes i and j are connected and $A_{ij} = 0$ when they are not.

We assume same interaction strength J between all interacting pairs of spins, so the Hamiltonian of the system can be written as

$$H = -J \sum_{i,j} A_{ij} \delta_{s_i, s_j}, \tag{1}$$

where δ is the Kronecker delta and $J > 0$ is the coupling constant (we consider only ferromagnetic interactions). Since only a difference in energies possesses a physical meaning, not energy values themselves, for $q = 2$ this Hamiltonian is equivalent to that of the well-known Ising model without an external field. Note that the discrete Potts model is not equivalent to the Ising model with more spin states (for example $s_i \in \{-1, 0, 1\}$). In fact for the Potts model, the only possible energy of interactions corresponds to matching or unmatching neighboring spins, while in the Ising model with higher spins such energy can possess more values. Different states in the Potts model can be treated as "orthogonal",[1] unlike different scalar values corresponding to the Ising model. We allow changes of both node states and links between nodes, although topology dynamics consists only of rewiring, keeping the number of links constant. Both spin and link dynamics follow the same principle as Metropolis algorithm, where changes are proposed and accepted or rejected based with probability $e^{-\beta \Delta H}$. During an update of a node, we either attempt changing the node state (as in the regular Potts model) or changing the other endpoint of incident edge. Edge dynamic is attempted with a fixed probability p, which is a system parameter. The system is treated as being in a constant temperature T, scaled in J/k_B units, meaning that $J = 1$ and $\beta = 1/T$.

In the simulations, we have used an asynchronous update rule, where each time step consists of N single updates of randomly chosen nodes (attempting to change either state or one of incident links). While the update rule can have a significant impact on the model dynamics, the Metropolis algorithm relies on single state changes to efficiently probe phase space. An alternative would be to use a synchronous approach. This would however mean either defeating purpose of Metropolis algorithm and random phase space search or discrepancy between probabilities of state change and energy difference if single transition probabilities are used. Such discrepancy may lead to artificial behaviors, such as oscillatory states in the Ising model [20]. Since our aim was to base system dynamics on the Hamiltonian, we used Metropolis algorithm and kept the asynchronous update as the only considered rule. The co-evolutionary dynamics is frequently considered as applying to social systems. In fact, humans do not act according to one common clock, and interactions between them – exchange of information and opinions, influencing each other – are not synchronized. Therefore in such a situation, the asynchronous updating is very well justified.

[1] If we use product of spins in the Hamiltonian (as in Ising model) $H = -J \sum_{i,j} s_i s_j$ but treat different states $s = 1, 2, 3, \ldots$ as orthogonal unit vectors in q dimensional space then the scalar product is equivalent to the δ_{s_i, s_j}.