Physica A 463 (2016) 103–110

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Breakdown of the modulational approximation in a multimode extension of the triplet interaction

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HIGHLIGHTS

- The three wave interaction in a multi-mode environment is studied.
- The analysis is not restricted to slow modulational approximations.
- Abrupt regime transitions arise as a result of the full analysis.

ARTICLE INFO

Article history:
Received 6 July 2016
Available online 19 July 2016

Keywords:
Nonlinear waves
Three wave interaction

ABSTRACT

The present work investigates the breakdown of the modulational approximation in a multimode extension of the three wave nonlinear interaction. The modulational approach provides a simplified framework to describe the triplet interaction in regimes where the carrier frequency is much larger than the modulational frequency. The approach is frequently stretched to its limits and beyond, and in the present work we argue that those limits can be in fact quite restrictive. At very small couplings we show that all modes exhibit slow amplitude modulations, but as the coupling increases a transition soon takes place and the modes jump to a new dynamical regime where none can be any longer seen as slowly modulated high-frequency harmonic carriers. Estimates for the critical coupling and relaxation times can be obtained with proper analysis of the most unstable triplet (Iorra et al., 2015).

1. Introduction

The slow modulational approximation to the dynamics of high-frequency carrier modes has been proven time and time again as a powerful technique to deal with systems involving the interaction of several degrees-of-freedom. Instead of describing the oscillatory modes at their short time or space scales, the modulational approach allows to obtain approximate governing equations for a smoother varying set of dynamical variables: the amplitudes and phases of the interacting modes [1,2].

Modulational techniques have been applied to a variety of physical settings, ranging from mechanical waves in solids and seismic media [3], to electromagnetic waves in plasmas, plasma accelerators and free-electron lasers [4–13]. In all cases, the needed condition for accuracy is that the mode interaction be weak enough that amplitudes and phases indeed change in a much longer scale than the high-frequency time scale and wavelength spatial scale of the carriers.

The modulational theory has been particularly successful in the study of three wave systems, where energy exchange involving three modes is possible if parametric instabilities are present. The wave triplet is a cornerstone in the study of

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http://dx.doi.org/10.1016/j.physa.2016.07.010
0378-4371/

nonlinear wave interaction and more complex interactive settings can be frequently understood with basis on three wave partitions [14].

Considering the importance of the three wave interaction, a recent work investigated the behaviour of the triplet dynamics as the coupling grows beyond the proper validity range for modulational approximations. It has been found that there exists indeed a critical coupling strength separating modulational and irregular regimes, where in the latter the amplitudes jump to much larger and much faster oscillations than in the former [15].

While an isolated triplet can well represent cases where resonant conditions favour only three modes at a time, it is frequently the case that a larger number of modes can simultaneously participate in the nonlinear interaction. Such is the case when a fundamental mode interacts with its own harmonics in a medium where the phase-velocity of the relevant waves is at least approximately constant. Here, owing to the linear quality of the associated dispersion relation, several intermingled triplets are excited from a pump and the isolated triplet approach can at most serve as an aid to understanding the larger and more complex system [16].

In the present work we thus focus on a multimode extension of the triplet interaction to address the corresponding behaviour as the nonlinear coupling between the various modes increases. The question to be examined here is basically whether or not a critical coupling strength is present defining a transition from a smoother to a less regular type of dynamics, similarly to what happens with an isolated triplet. As we shall see, a transition will be indeed identified and argued to be of relevance to nonlinear wave fields with cubic nonlinearities in the corresponding Lagrangian or Hamiltonian functions. Not only that, but we shall also see that in the case where a large number of modes are actively involved in the dynamics, the critical coupling becomes unsuspectingly small.

2. Full Lagrangian, its parameters and initial settings

We start with the Lagrangian for the multimode interaction of a collection of N modes in the form

$$L = \frac{1}{2}\sum_{j=1}^{N}\left(\dot{x}_j^2 - {\omega_j}^2 x_j^2\right) - \varepsilon \sum_{\substack{1\le i<j<k \\ i+j-k=0}} g_{i,j,k}\, x_i x_j x_k. \tag{1}$$

The natural frequencies of the N modes are denoted by ω_j with $j = 1, 2, \ldots, N$, and among the ordered summation over unrepeated combinations of indexes i, j, k, one selects only those for which the indicated matchings can occur. The modal integer indexes i, j, k mimic wave vectors in a dimensionless form, and with this summation rule we intend to comply with the general view on the triplet interaction that each triplet is formed by different modes under wave vector matching conditions. Whenever frequency matching is satisfied as well, strong resonant coupling occurs.

The frequencies ω_i shall be taken to be linearly homogeneous on index "i", which represents the case where mode frequencies are solely determined by wave vectors arising from second order derivatives operating on a virtually associated spatial coordinate axis. In this case frequency matching poses no further restriction than the wave vector matching, and several triplets can be simultaneously excited, which adhere to the focus of the present analysis as stated earlier. This sort of physical modelling is of relevance in the collinear interaction of broadband optical and non optical modes in nonlinear media [17,18].

Parameter ε measures the intensity of the triplet coupling represented by the product $x_i x_j x_k$, and the form factor g is chosen to guarantee circumscription of the dynamics to finite regions of the phase-space. One notes indeed that the cubic product has no definite sign and, if alone, can therefore generate unconfined trajectories escaping to infinity. The purpose of the form factor g is thus to confine the dynamics to finite regions of the configuration space. In the present paper we choose $g_{i,j,k} = g(x_i, x_j, x_k) = \mathrm{e}^{-(x_i^2+x_j^2+x_k^2)/\sigma^2}$ with a relatively large value of σ. As the quadratic sum exceeds the value of σ^2 the interaction is automatically switched off with $g \to 0$ and the remaining harmonic potential prevents the mode coordinates from shooting to infinity. On the other hand, when mode excursions are always much smaller than σ, then $g \to 1$ and the coupling acts as a small perturbation producing the slow modulational changes on the modal amplitude and phases; this is where one encounters the slow modulational regime. In a sense, form factor g incorporates higher-order nonlinearities saturating the triplet interaction. We have also tried polynomial forms of confinement, which produce essentially the same results, although at larger computational costs. The exponential therefore turns up as the preferred choice here.

All modes start off random noise level amplitude, $|x_i| \le 0.01$, except for mode $i = N/2$ which acts as the initial pump. Unless otherwise stated, we take $x_{N/2} = 1$. We place the pump midway of the spectrum in order to examine bilateral energy distribution as the system evolves. This way one can examine in a very symmetrical way the inverse cascade down to mode $j = 1$ and the ensuing direct energy cascade up to mode $j = N$ as time evolves. The assumption that no mode beyond the Nth is excited, can be laid on slightly firmer physical grounds if one adds strong dissipative rates beyond mode N. With strong dissipation added this way, every mode outside the considered group would be fully quenched and energy would remain confined in the initial collection.

Finally, we normalize time such that $\omega_{i=N/2} = 1$. Under these conditions we have $\omega_i = 2i/N$, with $\omega_1 = 2/N$ and $\omega_N = 2$ in particular.

The particular form for the nonlinear potential chosen here allows to access the role of the three wave interaction. Examining the limit of small amplitudes $x^2 \ll \sigma^2$ for illustrative sake, we write down the Euler–Lagrange equation for