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Fluctuations of a surface relaxation model in interacting scale free networks

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ABSTRACT

Isolated complex networks have been studied deeply in the last decades due to the fact that many real systems can be modeled using these types of structures. However, it is well known that the behavior of a system not only depends on itself, but usually also depends on the dynamics of other structures. For this reason, interacting complex networks and the processes developed on them have been the focus of study of many researches in the last years. One of the most studied subjects in this type of structures is the Synchronization problem, which is important in a wide variety of processes in real systems. In this manuscript we study the synchronization of two interacting scale-free networks, in which each node has ke dependency links with different nodes in the other network. We map the synchronization problem with an interface growth, by studying the fluctuations in the steady state of a scalar field defined in both networks.

We find that as ke slightly increases from $ke = 0$, there is a really significant decreasing in the fluctuations of the system. However, this considerable improvement takes place mainly for small values of ke, when the interaction between networks becomes stronger there is only a slight change in the fluctuations. We characterize how the dispersion of the scalar field depends on the internal degree, and we show that a combination between the decreasing of this dispersion and the integer nature of our growth model are the responsible for the behavior of the fluctuations with ke.

1. Introduction

In the last decades the study of complex networks has attracted the attention of many researchers because many real processes evolve on these types of structures. In early stages of these studies researchers were focused on processes that develop on isolated networks, however, systems, in general, are not completely isolated, but interacting with other systems instead. These types of interacting systems, which are a special case of the class called Networks of Networks (NoN), are composed of internal and external connections. NoN structures were successfully used to understand epidemic spreading [1–5], cascade of failures [4–9], diffusion [4,5,10,11] and synchronization [4,12–15].

Synchronization phenomena is a relevant subject in many areas, such as in neurobiology [15–19], animal behavior [20–22], power-grid networks [23–25] and so forth. In a relatively recent approach, synchronization problems in complex networks are associated to the fluctuations of a scalar h defined over the system [14,26–39]. This scalar field is a measure of

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the amount of load that a node has to manage. For example, in the problem of queuing networks, the load is proportional to the waiting time that a node needs to complete his task. The load in a node must be reduced in order to avoid increasing the waiting time by distributing efficiently the loads and thus improving the synchronization. In this approach the fluctuations are given by

$$W = \sqrt{\left\{ \frac{1}{N} \sum_{i=1}^{N} (h_i - \langle h \rangle)^2 \right\}}, \tag{1}$$

where h_i is the load of node i, $\langle h \rangle$ is the average value of the load over the network, N is the system size, and $\{\}$ is the statistical average. In the steady state the fluctuations reach a constant value $W \equiv W_s$, which depends on the topology of the network. This type of process was studied in networks with different topologies, but in the last few years many researches have focused on Scale-Free (SF) networks because they are obiquous in many real systems. These kinds of networks are characterized by a degree distribution $P(k) \sim k^{-\lambda}$, where $P(k)$ is the probability that a node has k internal links and λ is the exponent of the power law distribution. In general, λ takes values between 2.5 and 3 in real SF networks.

One of the most studied models of growth interface is the Family model [14,35–40], which is a surface relaxation model (SRM). In this model, at each time-step a node is randomly chosen, and the node with the lowest amount of load or 'height' between the selected node and its neighbors increases its load in one unit. In isolated SF networks with exponent $\lambda < 3$ it was found that the dependence of W_s with the system size N has a crossover between two different behaviors at a characteristic size N_0: for $N < N_0$, $W_s \sim \log N$, and $W_s \sim$ constant for $N > N_0$ [35]. Thus in the last regime the system is scalable, i.e. increasing the system size does not affect the fluctuations. In a more recent work [14] the authors studied the SRM in two interacting SF networks, in which a fraction q ($0 \leq q \leq 1$) of nodes in each network is connected one by one through bidirectional external links, allowing diffusion from one network to another. They found that the synchronization improves as q increases and reaches an improvement of 40% for $q = 1$. In real systems however, nodes can have more than one external connection with nodes in the other networks, which implies a stronger interaction between the systems. This strong interaction may affect the processes that develop on structures of this type. In this work we are interested in understanding how the strong interaction between networks affects the synchronization of the system. For this purpose we study the SRM model in two SF networks in which each node has *ke* external connections. In this study we only use stochastic simulations due to the fact that the heterogeneity of the SF networks and the lack of geometrical direction makes difficult any theoretical approach [36].

2. Model

We build two SF networks A_i ($i = 1, 2$) using the Molloy–Reed Algorithm [41], avoiding multiple and self connections, and we use a minimum degree $k_{\min} = 2$ to ensure that each network has a single component [42]. In both networks every node j, with $j = 1, \ldots, N$, has k_j^i internal connections with nodes in the same network and *ke* external connections with nodes in the other network. By simplicity, we consider the same number of external connections for all nodes. We denote by v_j^i and b_j^i the set of internal and external neighboring nodes respectively of node j from network A_i. We chose as initial condition all the scalar fields h_j^i randomly distributed in the interval [0, 1].

At each time step a node j in one of the networks A_i, with $i = 1, 2$, is randomly chosen and receives a load unit. Then:

(1) The load diffuses to the node m, which is the one with the smaller load in the set $\{j, v_j^i\}$. We denote this process as the first internal diffusion.
(2) If h_m^i is smaller than all the heights in the set b_m^i, then the load is deposited in m and $h_m^i = h_m^i + 1$. (Color green in Fig. 1.) Otherwise the load diffuses to the node with the smaller height in the set $\{b_m^i\}$. We denote this process as external diffusion.
(3) If an external diffusion takes place, step (1) is repeated and, after a second internal diffusion, the load is deposited in a node n in the network A_l with $l \neq i$ (color red in Fig. 1). Then $h_n^l = h_n^l + 1$.

3. Results and discussions

For the simulations we build two SF networks with the same exponents $\lambda = 2.6$ and sizes $N = 3 \times 10^5$ to ensure that the system is in the scalable regime [35]. As the two networks used here have the same exponent λ and same system size N, the fluctuations W_s^i on each network will be in average the same, thus by simplicity we drop the index i. In Fig. 2 we plot the square fluctuations in the steady state of each network W_s^2 as a function of the external connection parameter *ke*. We can observe that the synchronization of the system improves as *ke* increases and that the fluctuations converge asymptotically to the optimal value $W_s^2(N)$, which corresponds to the case $ke = N$. The reduction in the fluctuations when more external connections are added is due to the fact that the overloaded nodes in one network have the possibility to diffuse their excess of load to nodes that possess lower levels of load in the other network. This external diffusion allows to synchronize nodes that have increasingly similar amounts of load. It is important to notice that for high interacting networks with $k_e = N$, the value $W_s^2(N)$ is independent of the exponent λ of the degree distribution because in this case the interaction between