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Exact solution to fractional logistic equation

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HIGHLIGHTS

- The logistic equation is generalized to include memory using the fractional calculus.
- The fractional logistic equation (FLE) is solved using an infinite-order linear representation.
- The linear representation of the FLE is exact and its solution involves the inversion of an infinite-order matrix.
- The technique is applicable to a large class of polynomial nonlinear fractional rate equations.

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ABSTRACT

The logistic equation is one of the most familiar nonlinear differential equations in the biological and social sciences. Herein we provide an exact solution to an extension of this equation to incorporate memory through the use of fractional derivatives in time. The solution to the fractional logistic equation (FLE) is obtained using the Carleman embedding technique that allows the nonlinear equation to be replaced by an infinite-order set of linear equations, which we then solve exactly. The formal series expansion for the initial value solution of the FLE is shown to be expressed in terms of a series of weighted Mittag-Leffler functions that reduces to the well known analytic solution in the limit where the fractional index for the derivative approaches unity. The numerical integration to the FLE provides an excellent fit to the analytic solution. We propose this approach as a general technique for solving a class of nonlinear fractional differential equations.

1. Introduction

The simplest ordinary differential equation that has found explanatory value in phenomena ranging from the decay of radioactive particles to the growth of populations is the rate equation, where the rate of change in the number of entities of interest $N(t)$ is determined by

$$\frac{dN(t)}{dt} = kN(t). \tag{1}$$

The solution to the rate equation is the simple exponential $N(t) = N(0)e^{kt}$, with $N(0)$ being the initial population. The number of remaining radioactive particles shrinks exponentially in time since the rate constant is negative ($k < 0$), whereas in animal population studies the rate constant is positive ($k > 0$) so that the population grows exponentially, as conjectured for human population by the eighteenth century cleric Malthus [1].

A fundamental assumption underlying Eq. (1) is that for the process being discussed time is a homogeneous passive quantity that indexes the changing events through the regular ticking of a clock. However this is not necessarily the case for

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complex phenomena wherein the time derivative is replaced by a fractional derivative to yield the fractional rate equation (FRE) [2]

$$D_t^\alpha [u(t)] = k^\alpha u(t). \tag{2}$$

where the rate of growth has been raised to a power of the order of the fractional derivative to retain dimensional consistency. We restrict the index to the domain $1 \geq \alpha > 0$ in order to retain the spirit of a growth process and we interpret the fractional time derivative in the Caputo sense. The Laplace transform of the Caputo fractional derivative is given by

$$LT\left\{D_t^\alpha [f(t)]; s\right\} \equiv \int_0^\infty e^{-st} D_t^\alpha [f(t)] = s^\alpha \widehat{f}(s) - s^{\alpha-1} f(0); \quad 1 \geq \alpha > 0 \tag{3}$$

and $\widehat{f}(s)$ is the Laplace transform of $f(t)$. The solution to the fractional rate equation in terms of the Laplace variable is

$$\widehat{u}(s) = \frac{s^{\alpha-1}}{s^\alpha + k} u(0) \tag{4}$$

whose inverse Laplace transform yields [3,2]

$$u(t) = E_\alpha(kt^\alpha)\, u(0) \tag{5}$$

where the Mittag-Leffler function (MLF) $E_\alpha(\cdot)$ is defined by the series

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha + 1)}. \tag{6}$$

In the limit $\alpha = 1$ Eq. (5) simplifies to the exponential solution to the linear rate equation.

The solutions to such FREs have provided excellent fits to empirical data in a wide range of applications from biological reactions in the making of tequila [4] and other bioengineering applications [5], to allometry relations characterizing the size-dependence of the time experienced by elephants, humming birds and all the animals in between Ref. [6].

A nonlinear growth equation was introduced into population dynamics by Verhulst [7] to quench the unbounded growth in human population proposed by Malthus [1] and mitigate the utterly dismal predictions of the fate of mankind entailed by Malthus' prediction. Verhulst [7] introduced the nonlinear term into the rate equation and obtained what subsequently became known as the logistic equation

$$\frac{du(t)}{dt} = ku(t)\,[1 - u(t)] \tag{7}$$

where the variable $u(t)$ is the population $N(t)$ normalized to its maximum attainable value N_{max}; $u(t) = N(t)/N_{max}$. The logistic equation is one of the few nonlinear equations that has a known exact closed form solution:

$$u(t) = \frac{u_0}{u_0 + (1 - u_0)e^{-kt}} \tag{8}$$

where the initial $(t = 0)$ state is $u_0 = N(0)/N_{max}$. In the asymptotic limit the normalized population approaches the value unity. This kind of saturation behavior has been used to explain all manner of complex phenomena where the initial exponential growth of a population is eventually curtailed by limited resources becoming exhausted.

The sigmoidal behavior of the solution to the logistic equation in addition to applications in ecology has been observed in the product concentration of autocatalytic reactions, which has been argued by some to play a major role in the process of life itself [8,9]. There has been increasing application of the logistic curve in medicine where it has been used to model the growth of tumors [10]. It has also been used to model the social dynamics of replacement technologies by Fisher and Pry [11] and the adaptability of society to innovation. Examples of the latter in the transportation of goods appear as, canals being replaced by railroads, which were ceded to intercoastal highway systems, that in turn were replaced by international airfreight, each in its turn being represented by its own sigmoidal growth [12].

The dual aspects of complexity, fractional growth and nonlinearly induced saturation, dovetail into the fractional logistic equation (FLE):

$$D_t^\alpha [u(t)] = k^\alpha u(t)\,[1 - u(t)]\,. \tag{9}$$

Of course one cannot use the Laplace transform method to directly solve such a nonlinear fractional dynamic equation and a number of creative approximation techniques have been devised for its solution [13–15]. Kowalski [16] pointed out that in 1932 Carleman [17], following the ideas of Poincaré and Fredholm, demonstrated that a nonlinear system of ordinary differential equations with polynomial nonlinearities can be reduced to an infinite-order system of linear differential equations, without approximation. One can then adopt their favorite method for solving the set of linear equations and extract the solution to the original nonlinear equation.

Herein we construct an exact solution to the FLE using the Carleman embedding technique. In Section 2 we demonstrate how to apply the technique to construct an infinite-order system of linear fractional differential equations equivalent to the FLE. The infinite-order linear system of equations is solved exactly using Laplace transforms and matrix methods to obtain a solution in terms of a weighted sum over MLF's. In Section 3 the analytic solution to the FLE is shown to yield the logistic form of Eq. (8) when $\alpha = 1$ and to agree with the direct numerical integration of the FLE for a variety of values of α. We draw some conclusions in Section 4.