Physica A 448 (2016) 205–215

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Critical value for contact processes on clusters of oriented bond percolation

Xiaofeng Xue

School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

HIGHLIGHTS

- We study contact processes on clusters of oriented bond percolation.
- We introduce the definitions of the critical values in the annealed case and quenched case.
- We show that the two critical values are equal.
- We show that the critical value is close to $1/dp$ for large dimension d, where p is the probability of open.

ARTICLE INFO

Article history:
Received 8 April 2015
Received in revised form 22 September 2015
Available online 2 January 2016

Keywords:
Contact process
Percolation
Open cluster
Critical value

ABSTRACT

In this paper we are concerned with contact processes on open clusters of oriented bond percolation in $\mathbb{Z}^d$, where the disease spreads along the direction of open edges. We show that the two critical values in the quenched and annealed cases are equal with probability one and are asymptotically equal to $(dp)^{-1}$ as the dimension d grows to infinity, where p is the probability that the edge is 'open'.

1. Introduction

In this paper we are concerned with contact processes on open clusters of oriented bond percolation in $\mathbb{Z}^d$. In our model, for any $x, y \in \mathbb{Z}^d$, there is a directed edge from x to y if and only if $y - x \in \{e_i\}_{1\leq i\leq d}$, where

$$e_i = (0, \ \ldots, \ 0, 1_{ith}, 0, \ \ldots, \ 0).$$

We denote by E_d the set of directed edges on $\mathbb{Z}^d$. $\{X_e\}_{e\in E_d}$ are independent and identically distributed random variables such that

$$P(X_e = 1) = 1 - P(X_e = 0) = p \in (0, 1).$$

Edge e is called 'open' if $X_e = 1$ or 'closed' if $X_e = 0$. We denote by $x \to y$ when the edge from x to y is open. After deleting all the closed edges, we obtain an oriented subgraph G of $\mathbb{Z}^d$, which our contact process will be defined on. Please note that G is a random graph depending on the values of $\{X_e\}_{e\in E_d}$.

E-mail address: xuexiaofeng@ucas.ac.cn.

http://dx.doi.org/10.1016/j.physa.2015.12.101
0378-4371/

The contact process $\{\eta_t\}_{t\geq 0}$ on G is a spin system with state space $\{0, 1\}^G$, which means that at each vertex, there is a spin with value 0 or 1. The flip rates of $\{\eta_t\}_{t\geq 0}$ are given by

$$c(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda \sum\limits_{y:y\to x} \eta(y) & \text{if } \eta(x) = 0 \end{cases}$$

for any $(x, \eta) \in G \times \{0, 1\}^G$, where $\lambda > 0$ is the infection rate. For more details about spin systems, please see Chapter 3 of Ref. [1].

Intuitively, the model describes the spread of an infectious disease along the direction of open edges. 1 and 0 represent the state 'infected' and 'healthy' respectively. An infected vertex waits for an exponential time with rate one to recover. For any healthy vertex x, if y is infected and the edge from y to x is open, then y infects x at rate λ.

In real life, diseases spreading along one direction are those traveling by rivers, such as dysentery, cholera, typhoid and so on. Closed edges represent the river courses which are too dry to carry the disease.

Recently, contact processes in random environments such as percolation model are a popular topic. Here we list some results in this field which inspire us. In Refs. [2,3], Chen and Yao prove that the complete convergence theorem holds for contact processes in two kinds of random environments on $\mathbb{Z}^d \times Z^+$, one of which is the percolation model. In Ref. [4], Chatterjee and Durrett show that contact processes on random graphs with power law degree distribution have critical value 0, which is not consistent with the estimation given by non-rigorous mean field approach. In Ref. [5], Peterson shows that the critical value of contact processes on complete graphs with random vertex-dependent infection rates is inversely proportional to the second moment of the weight of a vertex. Inspired by [5], Xue studies the contact process with i.i.d random vertex weights ρ on the oriented lattice Z^d in Ref. [6] and shows that the critical value is close to $1/dE\rho^2$ for large d. Xue also studies contact processes with random vertex weights ρ on general regular lattices in Ref. [7] and shows that $1/E\rho^2$ is a lower bound of the critical value in the sense of the fluid limit.

2. Main results

We need introduce some notations before stating the main problem we are concerned with. In later sections, we denote by P_λ^G the probability measure of the contact process with infection rate λ on a given graph G, which is called the quenched measure. We denote by E_λ^G the expectation with respect to P_λ^G. Note that G depends on the values of $\{X_e\}_{e\in E^d}$, which leads to the following notations. We assume that $\{X_e\}_{e\in E^d}$ are defined on the product measurable space $(\{0, 1\}^{E_d}, \mathcal{F}_d, \mathbb{P}_{d,p})$ (see Section 1.3 of Ref. [8]), where p is the probability of 'open'. We denote by $\mathbb{E}_{d,p}$ the expectation with respect to $\mathbb{P}_{d,p}$. For any $\omega \in \{0, 1\}^{E_d}$, we denote by $G(\omega)$ the random graph of oriented percolation depending on $\{X_e(\omega)\}_{e\in E_d}$. We define

$$P_{\lambda,d,p}(\cdot) = \mathbb{E}_{d,p}\big[P_\lambda^{G(\omega)}(\cdot)\big],$$

which is called the annealed measure. We denote by $E_{\lambda,d,p}$ the expectation with respect to $P_{\lambda,d,p}$.

In later sections, we write η_t as η_t^A when

$$\{x : \eta_0(x) = 1\} = A.$$

If all vertices are infected at the beginning, then we omit the superscript.

Since the contact process is attractive (see the definition of attractive in Chapter 3 of Ref. [1]), it is easy to see that $P_\lambda^G(\eta_t(x) = 1)$ is decreasing with t for any $x \in G$ and so does $P_{\lambda,d,p}(\eta_t(\mathbf{0}) = 1)$, where $\mathbf{0}$ is the origin of $\mathbb{Z}^d$. Furthermore, according to the basic coupling of spin systems (See Chapter 3 of Ref. [1]), if $\lambda_1 \geq \lambda_2$, then

$$P_{\lambda_1}^G(\eta_t(x) = 1) \geq P_{\lambda_2}^G(\eta_t(x) = 1)$$

and

$$P_{\lambda_1,d,p}(\eta_t(\mathbf{0}) = 1) \geq P_{\lambda_2,d,p}(\eta_t(\mathbf{0}) = 1).$$

As a result, the definitions of the following critical values are reasonable. For $d \geq 1$, $p \in (0, 1)$ and random graph G with respect to $\{X_e\}_{e\in E^d}$, we define

$$\lambda_c(d, p) = \sup\{\lambda : \lim_{t\to+\infty} P_{\lambda,d,p}(\eta_t(\mathbf{0}) = 1) = 0\} \tag{2.1}$$

and

$$\widehat{\lambda_c}(G) = \sup\{\lambda : \forall x \in G, \lim_{t\to+\infty} P_\lambda^G(\eta_t(x) = 1) = 0\}. \tag{2.2}$$

According to the translation invariance of our model, $P_{\lambda,d,p}(\eta_t(x) = 1)$ does not rely on the choice of x. However, the contact process on a given G is not symmetric for each vertex, which explains the difference between the two definitions.

The main problem we are concerned with is the estimation of $\lambda_c(d, p)$ and $\widehat{\lambda_c}(G)$. The following theorem is our main result.