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An entropical characterization for complex systems becoming out of control

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HIGHLIGHTS

- We study complex systems whose components have a hierarchical sub-structure.
- The system is partitioned into cells to which a generalized entropy S is associated.
- The different regimes of S define an entropical classification of the components.
- This classification is independent of the observer's criteria.
- The normalized S can be associated to an evolution in the absence of control.

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ABSTRACT

General properties of N-dimensional multi-component or many-particle systems exhibiting self-similar hierarchical structure are presented. The entire system is partitioned into cells, which have an associated generalized entropy $S(D)$ that is shown to be a universal function of the fractal dimension D of the configurations, exhibiting self-similarity properties which are independent of the dimensionality N. This provides a general way to classify the components of the system according to entropical reasons, independently of the observer's criteria. For certain complex systems, the normalized $S(D)$ may also be associated with the large time stationary profile of the fractal density distribution in the absence of external fields (or control).

1. Introduction

Multi-component, strongly correlated systems, often exhibit non-linear behavior at the microscale leading to emergent phenomena at the macroscale. As P.W. Anderson stated back in 1972, it often happens that *"the whole becomes not only more but very different from the sum of its parts"* [1]. Fingerprints of such emergent phenomena can be identified in hierarchical behavior [2,3] or constraints [4,5], sometimes associated with fractal behavior [6].

Hierarchical behavior exhibiting self-similarity has been identified in physical, social, biological and technological systems [7]. Employing theoretical tools, such as the *singularity spectrum* or its equivalent, *multiscaling exponents* (via a Legendre's transformation), etc., fractal analysis has been applied to geophysics, medical imaging, market analysis, voice recognition, solid state physics, etc. (see e.g. Refs. [8–12]). Recently, high quality spatio-temporal fractal behavior in connection with built-up areas in planar embeddings [13] has been unraveled, whose diversity of fractal dimensions covered

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the entire [0, 2] dimensionality spectrum, reflecting the presence of self-organizing principles which strongly constrain the spatial layout of the urban landscape. Nevertheless, the ideas developed in Ref. [13] are by no means restricted to the study of Urban Dynamics. This is the starting point of the present work, which introduces a general framework to describe the behavior of multi-component complex systems constrained to exhibit fractal behavior in a space of arbitrary dimensionality N. New properties, ideas and concepts applicable to the study of Complex Systems arise from this generalization, which will be developed as follows.

Section 2 outlines the classes of complex systems studied, the hierarchies associated to its components, and the central role that the concept of entropy will play in this article. Section 3 shows many estimates that enable us to numerically analyze and compute the entropy function for $N > 2$. Section 4 introduces self-similar properties that are used to carry out a pattern analysis of the components. All the ideas developed in the previous sections will be finally applied to study Complex Systems evolving *in the absence of control* in Section 5.

2. Classes of complex systems studied and entropy function

2.1

It is assumed that the complex systems of this article will be constituted of many interacting components. Every component will have a kind of hierarchical internal sub-structure [14], whose associated hierarchy will be indexed by some number D, ranging between a pair of minimum and maximum values, say $D = 0$ and $D = N > 0$, respectively. Thus, the hierarchies will be ordered and the components of the system could be classified according to D. Even for many systems, the hierarchy D may not only indicate something about the structure of the component, but it could also define a specific type of dynamics (or *pattern* of behavior) that the component follows, as it interacts with the rest.

Physics often needs to define ideal models ranging from say, *spherical horses in vacuum* [15] to symplectic structures of dynamical laws of any kind (relativistic, Newtonian, quantum, etc.). Clearly, all these structures cannot be applied to complex systems. In contrast, the systems of this nature seem to follow so complex dynamical laws that a good idea could be – perhaps – setting our point of reference at the opposite side, i.e. at a kind of *worst possible ideal scenario*, which will be out-of-control, in the sense that our knowledge about what is inside each component is minimal. This scenario may be represented by a distribution function $\rho_*(D)$,[1] so that the number of components with hierarchy index D will be proportional to our ignorance of the particular dynamics represented by D. Given that the physical notion of entropy corresponds to a measure of how much one does not know about something, we shall assume that $\rho_*(D)$ is given by:

$$\rho_*(D) = \frac{S(D)}{\int_0^N S(x)\,\mathrm{d}x}, \tag{1}$$

where $S(D)$ is an entropy function in a generalized sense. Of course, how one could precisely define a hierarchy index D and its entropy function $S(D)$ for a general complex system, is a difficult question to answer. An alternative to this can be found in the article [13].

The article is about Urban Dynamics and Complexity. In it, a metropolitan area is visualized as composed of many cells (squares of 1 km side), each one indexed by a *number* associated to the way the structure of the built-up area inside of it is hierarchically organized. The *number* is the fractal dimension D of the built-up area of the cell. Similarly, throughout the present article, it will be assumed that the fractal dimension D indexes the hierarchies and their associated dynamics of the components. This is just one of the reasons why the letter D was chosen at the beginning of this section.

The second reason has to do with Fig. 1, which was not included in Ref. [13]. Every red curve corresponds to the plot of an entropy function $S(D)$ defined (in Refs. [13]) as the log-number of the possible cell built-up area configurations, having fractal dimension equal to D. The successive proximity of the distribution of cells $\rho(D)$ to $\rho_*(D) = S(D)/\int_0^2 S(x)\,\mathrm{d}x$ ($N = 2$, see Eq. (1)) is a subtle fact that is worth mentioning. The city under consideration in the article is Lisbon, for which there is a well-documented urban sprawl together with a number of urbanistic problems, due to either weak regulations or vague governmental controls in urban planning (see also Ref. [16]) during a period of more than 40 years. Clearly, *urban sprawl* is an out-of-control situation and I suggest that Fig. 1 could be seen as a *signature* of this, due to the notion that what cannot be controlled is precisely what is ignored. Then, given that $S(D)$ measures ignorance, it could be interpreted as proportional to the maximum degree of uncontrollability of the cells having dimension D. Thus, $\rho_*(D)$ represents the ideal worst possible scenario mentioned above, in which the amount of cells of some type D is proportional to how much out of control they can be. In connection with the title of the present work, we will get back to this issue in Section 5.

2.2

First of all, it is important to understand why the characterization presented in Ref. [13] in terms of counting the number of cell fractal configurations, is an idea very plausible to apply to many other higher dimensional complex systems, having – in principle – nothing to do with Urban Dynamics.

[1] I.e. satisfying $\int_0^N \rho_*(D)\,\mathrm{d}D = 1$ and such that $\rho_*(D)\mathrm{d}D$ is the fraction of components with a hierarchy index equal to D.