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Remarks on Fenchel–Moreau conjugate in the setting of consumer theory☆

George H.M. Cunha, Michel A.C. Oliveira, Wilfredo Sosa*

Graduate School in Economics, Catholic University of Brasilia, SGAN 916 – Modulo B, 70790-000 Brasilia, DF, Brazil

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Abstract

In this manuscript we consider the conjugate notion focused from consumer theory as an interesting tool. According to us, conjugate notion remained undeveloped in economic theory because Fenchel's conjugate notion was introduced exclusively for proper convex lower semi continuous functions and convexity assumption is not natural in economic theory. Nevertheless, we introduce necessary and sufficient optimality conditions for consumer problem. Also, we consider a particular version of Fenchel–Moreau conjugate notion, for lower semi continuous functions recently introduced in the literature as a generalization of Fenchel's conjugate. Finally, we adapt it to closed functions in order to define a convex dual problem for consumer problem.

Keywords: Convex analysis; Fenchel's conjugate; Economic theory

JEL classification: C02; C61; D11

Resumo

Neste artigo consideramos a noção de conjugada focalizada na teoria do consumidor como uma ferramenta interessante. Segundo nosso entendimento, a noção de conjugada ainda não foi desenvolvida na teoria econômica, porque a Conjugada de Fenchel foi introduzida exclusivamente para funções próprias, convexas semi continuas inferiormente e a convexidade não é uma hipóteses natural na teoria econômica. Mesmo assim, introduzimos condições necessárias e suficientes de otimalidade para o problema do consumidor. Finalmente, considerando uma versão particular da Conjugada de Fenchel-Moreau para funções semi continuas inferiores, recentemente introduzida na literatura como uma generalização da conjugada de Fenchel, e à adaptamos para as funções fechadas com o objetivo de definir um problema dual convexo para o problema do consumidor.

Palavras-chave: Analise convexa; Conjugação de Fenchel; Teoria Econômica

* Corresponding author.
E-mail addresses: george@ucb.br (G.H.M. Cunha), michel@ucb.br (M.A.C. Oliveira), sosa@ucb.br (W. Sosa).
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1. Preliminaries

Respect to notions coming from convex analysis, used here, were adopted those from Rockafellar (1974) (convexity, concavity, inner product, lower (upper) semi continuity, proper functions, etc.).

It is well known that Fenchel's conjugate plays an important role for instance in Functional Analysis, Convex Analysis and Optimization theory. From mathematical point is view, there are a lot of works in the literature. For example in Rockafellar (1974) make a systematic study for Convex Analysis, the series of works (Singer, 1986, 1989, 1991) treat the Duality Theory for Optimization Theory, in Martinez Legaz (2005) treat generalized convex duality and its economical applications, etc.

In Production Theory, "revenue minus production costs generate profit firm", so Fenchel's conjugate (Fenchel, 1949) of a proper convex lower semi continuous function, which represent production cost of a firm is nothing else that maximum profile (see Section 2). As this natural interpretation of Fenchel's conjugate notion, there are many properties of Fenchel's conjugate, for considering it as an interesting tool in Economic Theory. For example, the involution property for proper convex lower semi continuous functions. This involution property say that the biconjugate of a proper convex lower semi continuous function is exactly the original function, because biconjugate is nothing else that closed convexification of original function. This property is very important in Convex Duality Theory, because the optimal value of dual problem of a convex problem, when it is generated by a proper convex lower semi continuous perturbed function, is exactly the optimal values of the original convex problem. When it occur, we say that there is no duality gap. Unfortunately, Fenchel's conjugate notion was introduced exclusively for proper convex lower semi continuous functions and convexity (or concavity) is no a natural assumption in Economic Theory.

Twenty one years after to Fenchel contribution, Moreau generalized Fenchel's conjugate (Moreau, 1970), but this involution property no hold in general and dual problem may be no convex. Recently (2011), was introduced a Fenchel–Moreau conjugate for lower semi continuous functions, where this involution holds and so we can again maintain the economical interpretation of Fenchel's conjugate. In particular, our work try to applied Fenchel–Moreau conjugate to consumer problem.

Firstly, in Section 2, we make a briefly introduction to Fenchel and Fenchel–Moreau conjugate and its importance in convex duality. we finished Section 2 introducing upper, lower closed functions. Here, the family of lower (upper) semi continuous functions are included strictly in the family of lower (upper) closed function. Moreover, Representation Theorem (Theorem I in Debreu et al., 1983, p. 108) establish that when $\mathbb{R}^n$ is completely ordered by the order $\preceq$we have that: If for any $x' \in \mathbb{R}^n$ the sets $\{x \in \mathbb{R}^n : x \preceq x'\}$ and $\{x \in \mathbb{R}^n : x' \preceq x\}$ are closed, there exists on $\mathbb{R}^n$ a continuous, real, order preserving function. We point out that there exists a family of closed real order preserving functions (closed function is such that it is lower and upper closed simultaneously). For this reason we work with upper (lower) closed functions.

Then, in Section 3 we establish son results in order to characterize the solution set of consumer problem using Fenchel–Moreau conjugate.

Finally, in Section 3.1 we build a dual problem for the consumer problem, adapting the conjugate for lower semi continuous functions introduced in Cotrina et al. (2011) to lower closed functions.

2. Fenchel and Fenchel–Moreau conjugate

In 1949, Fenchel introduced the conjugate notion for convex and lower semi continuous functions based on the well known fact that many inequalities used in functional analysis (such as Minkowski, Jensen, and Young) may be considered as a consequence of the convexity of a pair of functions, which Fenchel called "conjugate functions." In a more precise formulation, Fenchel's result is the following: To each proper convex and lower semi continuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, there corresponds a function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with the same properties of f, such that

$$\langle x, y\rangle \leq f(x) + f^*(y),$$

for all x and y in $\mathbb{R}^n$. Here, functions f and f^* are called conjugate functions, and f^* is defined as follows:

$$f^*(y) := \sup\{\langle y, x\rangle - f(x) : x \in \mathbb{R}^n\}.$$

From economical viewpoint, taking f as a production cost, $\overline{y}$ as a production supply and $\overline{p}$ as a vector price. Here, conjugate function f^* represent maximum profile in the production. Unfortunately, these economical