Differential Geometry and its Applications 61 (2018) 82–96

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Symplectic slice for subgroup actions

Marine Fontaine

School of Mathematics, University of Manchester, M13 9PL Manchester, UK

A R T I C L E   I N F O

Article history:
Received 29 April 2018
Received in revised form 21 August 2018
Available online xxxx
Communicated by T. Ratiu

Keywords:
Symplectic slice
Hamiltonian dynamics
MGS normal form

A B S T R A C T

Given a symplectic manifold endowed with a proper Hamiltonian action of a Lie group, we consider the action induced by a Lie subgroup. We propose a construction for two compatible Witt–Artin decompositions of the tangent space of the manifold, one relative to the action of the big group and one relative to the action of the subgroup. In particular, we provide an explicit relation between the respective symplectic slices.

1. Introduction

Stability properties and bifurcations of relative equilibria is determined by a method developed by Krupa [1], which states that the dynamics of an equivariant vector field in a neighbourhood of a group orbit is entirely governed by its transverse dynamics. This result uses the so-called *slice coordinates* introduced by Palais [2]. Krupa first proves this result for compact Lie group actions and Fiedler et al. [3] extend it to proper Lie group actions. The Hamiltonian analogue is studied by Mielke [4] and Guillemin and Sternberg [5], as well as Roberts and de Sousa Dias [6] and Roberts et al. [7]. By "transverse dynamics" we mean that the vector field in question can be split into two parts: one part is defined along the tangent space of the group orbit and the other part belongs to a choice of normal subspace. In Hamiltonian systems the flow of a Hamiltonian vector field with a fixed initial condition is confined to a level set of the momentum map, reflecting the conservation of momentum. The choice of normal subspace is therefore more restrictive than for general dynamical systems. Before giving its explicit form we introduce some terminologies: given a symplectic manifold (M, ω) acted on by a Lie group G, the action is called *canonical* if it is smooth and it preserves the symplectic form ω. For any element $x \in \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of G we denote by x_M the vector field on M generated by the action. A canonical action is *Hamiltonian* if there exists a momentum map $\Phi_G : M \to \mathfrak{g}^*$ defined by the relation $\iota_{x_M}\omega = d\langle \Phi_G(\cdot), x\rangle$ for every $x \in \mathfrak{g}$. Here $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$.

E-mail address: marine.fontaine.math@gmail.com.

https://doi.org/10.1016/j.difgeo.2018.08.005
0926-2245/© 2018 Published by Elsevier B.V.

Definition 1.1. Assume that G acts properly and canonically on a symplectic manifold (M, ω). In addition we assume that the action is Hamiltonian and that the associated momentum map $\Phi_G : M \to \mathfrak{g}^*$ is equivariant with respect to the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$. The quadruple (M, ω, G, Φ_G) is called a *Hamiltonian G-manifold*.

We fix a Hamiltonian G-manifold (M, ω, G, Φ_G) and a point $m \in M$ with momentum $\mu = \Phi_G(m)$. The corresponding stabilizers for the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$ are denoted G_m and G_μ respectively. Their respective Lie algebras are denoted $\mathfrak{g}_m$ and $\mathfrak{g}_\mu$. Let $G \cdot m = \{g \cdot m \mid g \in G\}$ be the *G-orbit* of $m \in M$ and denote by $\mathfrak{g} \cdot m$ its tangent space at m. Elements of $\mathfrak{g} \cdot m$ are vectors of the form $x_M(m) := \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \exp(tx) \cdot m$, where $x \in \mathfrak{g}$ and $\exp : \mathfrak{g} \to G$ is the group exponential. A *symplectic slice* N_1 at m is a G_m-invariant subspace of $(T_m M, \omega(m))$ defined by

$$N_1 := \ker(\mathrm{D}\Phi_G(m))/\mathfrak{g}_\mu \cdot m, \tag{1.1}$$

where $\mathrm{D}\Phi_G(m)$ is the differential of Φ_G at m. It is endowed with a symplectic structure ω_{N_1} coming from $\omega(m)$, and a linear Hamiltonian action of G_m that makes it a Hamiltonian G_m-space. This subspace is of particular interest for the study of stability, persistence and bifurcations of relative equilibria (cf. Patrick et al. [8], Lerman and Singer [9], Ortega and Ratiu [10], as well as Montaldi and Rodríguez-Olmos [11]). The construction of a symplectic slice is based on a *Witt–Artin decomposition* (relative to the G-action) of $T_m M$ i.e. a decomposition into four G_m-invariant subspaces:

$$T_m M = T_0 \oplus T_1 \oplus N_0 \oplus N_1 \tag{1.2}$$

with respect to which the skew-symmetric matrix associated to $\omega(m)$ has a specific normal form. The part $T_0 \oplus T_1 = \mathfrak{g} \cdot m$ corresponds to the directions tangent to $G \cdot m$ whereas the part $N_0 \oplus N_1$ is a choice of normal subspaces such that $T_0 \oplus N_1 = \ker(\mathrm{D}\Phi_G(m))$. This decomposition first appears in Witt [12] for symmetric bilinear forms. The construction involves choices, namely the subspaces T_1, N_0 and N_1.

Let $H \subset G$ be a Lie subgroup with Lie algebra $\mathfrak{h}$ and inclusion map $i_\mathfrak{h} : \mathfrak{h} \hookrightarrow \mathfrak{g}$. The dual map $\mathfrak{i}^*_\mathfrak{h} : \mathfrak{g}^* \to \mathfrak{h}^*$ is given by $i^*_\mathfrak{h}(\lambda) = \lambda|_\mathfrak{h}$ which is the restriction of the linear form λ to the subalgebra $\mathfrak{h}$. Note that by definition, the projection $i^*_\mathfrak{h} : \mathfrak{g}^* \to \mathfrak{h}^*$ is H-equivariant. As the action of H on M is still Hamiltonian, it admits a momentum map $\Phi_H : M \to \mathfrak{h}^*$ given by $\Phi_H = i^*_\mathfrak{h} \circ \Phi_G$. Therefore (M, ω, H, Φ_H) is a Hamiltonian H-manifold and we call Φ_H the *induced momentum map*. In this case, we can also consider a Witt–Artin decomposition of $T_m M$ relative to the H-action:

$$T_m M = \widetilde{T}_0 \oplus \widetilde{T}_1 \oplus \widetilde{N}_0 \oplus \widetilde{N}_1. \tag{1.3}$$

In particular, the H_m-invariant subspace $\widetilde{N}_1$ is a symplectic slice for the H-action. It is chosen such that

$$\widetilde{N}_1 := \ker(\mathrm{D}\Phi_H(m))/\mathfrak{h}_\alpha \cdot m, \tag{1.4}$$

where $\alpha := \Phi_H(m)$ is the restriction of the linear form $\mu \in \mathfrak{g}^*$ to $\mathfrak{h}$. In general two arbitrary decompositions (1.2) and (1.3) cannot be compared.

In the study of explicit symmetry breaking phenomenons, Hamiltonian equations are perturbed in a way that the symmetry group G breaks into one of its subgroup H. This phenomenon is studied by many authors. References for the non-Hamiltonian case are for instance Lauterbach [13] or Chillingworth and Lauterbach [14]. Some aspects of the Hamiltonian case are studied in Ambrosetti et al. [16], Grabsi et al. [17], Gay-Balmaz and Tronci [15] or Fontaine [18]. The stability properties of the perturbed system rely on a symplectic slice relative to the H-action on M, which is “bigger” than a slice relative to the G-action. This