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Conformally flat hypersurfaces with constant scalar curvature

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ABSTRACT

We classify the conformally flat hypersurfaces $f\colon M^3 \to \mathbb{R}^4$ with three distinct principal curvatures and constant scalar curvature.

A well-known result by Cartan [3] states that a Euclidean hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ of dimension $n \geq 4$ is conformally flat if and only if f has a principal curvature whose multiplicity is at least $n-1$. Geometrically, a generic conformally flat Euclidean hypersurface of dimension $n \geq 4$ is the envelope of a one-parameter congruence of hyperspheres. Recall that a Riemannian manifold M^n is *conformally flat* if each point of M^n has an open neighborhood that is conformally diffeomorphic to an open subset of $\mathbb{R}^n$.

On the other hand, by a theorem due to do Carmo and Dajczer [2], if $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, has a principal curvature λ with multiplicity $n-1$ and the simple principal curvature μ is such that $\mu = \rho(\lambda)$ for some smooth function ρ, then f is a rotation hypersurface over a plane curve. In particular, a conformally flat hypersurface of dimension $n \geq 4$ with no umbilical points having a constant symmetric function of the principal curvatures is a rotation hypersurface over a plane curve that is determined by an ordinary differential equation. This is also the case for conformally flat hypersurfaces of dimension three in $\mathbb{R}^4$ having a principal curvature with multiplicity two everywhere.

Cartan himself observed that having a principal curvature with multiplicity at least two everywhere is no longer a necessary condition for a hypersurface $f\colon M^3 \to \mathbb{R}^4$ to be conformally flat. More recently, Hertrich-Jeromin [7] showed that conformally flat hypersurfaces in $\mathbb{R}^4$ with three distinct principal curvatures admit locally principal coordinates (u_1, u_2, u_3) such that the induced metric $ds^2 = \sum_{i=1}^3 v_i^2 du_i^2$ satisfies the Guichard condition, say, $v_2^2 = v_1^2 + v_3^2$. A recent improvement of that result by S. Canevari and the second author [1], characterizing conformally flat hypersurfaces in $\mathbb{R}^4$ with three distinct principal curvatures by the existence of such principal coordinate systems satisfying some additional conditions (see Theorem 3 below), is the starting point for the main theorem of this paper.

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It was shown by Defever [4] that if $f\colon M^3 \to \mathbb{R}^4$ is a conformally flat hypersurface with three distinct principal curvatures and constant Gauss–Kronecker curvature K, then K must be zero, that is, one of the principal curvatures of f must vanish everywhere. In this case, it is well-known (see Proposition 14 in [6]) that $f(M^3)$ is an open subset of either a cylinder over a surface with constant Gauss curvature in $\mathbb{R}^3$ or a cone over a surface with constant Gauss curvature in $\mathbb{S}^3 \subset \mathbb{R}^4$. He also proved in [5] that a conformally flat hypersurface with three distinct principal curvatures and constant mean curvature must be minimal. It was recently shown in [6] that there exists precisely a one-parameter family of minimal conformally flat hypersurfaces in $\mathbb{R}^4$ with three distinct principal curvatures.

In this paper we study conformally flat hypersurfaces in $\mathbb{R}^4$ with three distinct principal curvatures and constant scalar curvature. Besides aiming at completing the classification of conformally flat hypersurfaces in $\mathbb{R}^4$ with a constant symmetric function of its principal curvatures, there was another strong motivation for investigating this class of hypersurfaces.

Namely, a Riemannian manifold M of dimension three is conformally flat and has constant scalar curvature if and only if it has harmonic curvature tensor, that is, the divergence d^* of its curvature tensor R satisfies $d^*R = 0$. Here R is regarded as an element of $\Omega^2(M, \Lambda^2 M)$, that is, as a two-form on M with values in $\Lambda^2 M$. In view of the second Bianchi identity $dR = 0$, where d is the exterior differentiation, this is equivalent to $\Delta R = (dd^* + d^*d)R = 0$.

In any dimension, the identity $d^*R = -d\mathrm{Ric}$ implies that a Riemannian manifold has harmonic curvature tensor if and only if its Ricci tensor Ric is a Codazzi tensor, that is,

$$\nabla_X \mathrm{Ric}\, Y - \mathrm{Ric}\, \nabla_X Y = \nabla_Y \mathrm{Ric}\, X - \mathrm{Ric}\, \nabla_Y X \tag{1}$$

for all $X, Y \in \mathfrak{X}(M)$. Thus, Riemannian manifolds with harmonic curvature tensor are, in particular, natural generalizations of Einstein manifolds and of Riemannian manifolds whose Ricci tensor is parallel, that is, those for which

$$\nabla_X \mathrm{Ric}\, Y - \mathrm{Ric}\, \nabla_X Y = 0$$

for all $X, Y \in \mathfrak{X}(M)$.

Hypersurfaces in space forms with harmonic curvature tensor and arbitrary dimension have been investigated by some authors (see [9] and the references therein) but a classification is far from being achieved. On the other hand, hypersurfaces with parallel Ricci tensor in any space form were classified by Reckziegel [10], after previous work by Ryan [11]. In particular, if $f\colon M^n \to \mathbb{R}^{n+1}$ is a hypersurface with parallel Ricci tensor, then either M^n is flat (and f has 0 as a principal curvature with multiplicity $n-1$), or $f(M^n)$ is an open subset of either a round hypersphere, a $(n-k)$-cylinder over a sphere $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ or a $(n-2)$-cylinder over a surface in $\mathbb{R}^3$ with constant nonzero Gauss curvature. Notice that only the last of such hypersurfaces is not Einstein.

In spite of the fact that the Ricci tensor being a Codazzi tensor is, a priori, a much weaker assumption than that of being parallel, it follows from Theorem 1 below that, for hypersurfaces $f\colon M^3 \to \mathbb{R}^4$ with three distinct principal curvatures, there exists no example satisfying the former condition but not the latter.

Theorem 1. *If $f\colon M^3 \to \mathbb{R}^4$ is a conformally flat hypersurface with three distinct principal curvatures and constant scalar curvature, then $f(M^3)$ is an open subset of a cylinder over a umbilic-free surface in $\mathbb{R}^3$ with constant nonzero Gauss curvature.*

As mentioned before, conformally flat hypersurfaces in $\mathbb{R}^4$ with constant scalar curvature having a principal curvature with multiplicity two are rotation hypersurfaces over certain plane curves, and these have been classified in [8]. Therefore, Theorem 1 completes the classification of conformally flat hypersurfaces in $\mathbb{R}^4$ with constant scalar curvature (or equivalently, hypersurfaces in $\mathbb{R}^4$ with harmonic curvature tensor).