Differential Geometry and its Applications 61 (2018) 170–196

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

On symplectic dynamics

Stéphane Tchuiaga

Department of Mathematics of the University of Buea, South West Region, Cameroon

ARTICLE INFO

Article history:
Received 22 October 2016
Received in revised form 14 December 2017
Available online xxxx
Communicated by M. Gotay

MSC:
53D35
53D05
57R52
53C21

Keywords:
Differential forms
Linear sections
Isotopies
Riemannian metric
Symplectic diffeomorphisms
Hofer-like geometries

ABSTRACT

This paper contributes to the foundation of various Hofer-like topologies of a closed symplectic manifold (M, ω). Here, considering an arbitrarily linear section of the natural projection of the space of closed 1-forms onto the first de Rham's group, we study various Hofer-like metrics [2,7]. The outcome is that different splitting of the space of closed 1-forms lead to similar "symplectic analogues" of some results from Hamiltonian dynamics: Without appealing to the positivity result of any symplectic displacement energy, we point out an impact of the L^∞-Hofer-like lengths in the investigation of the symplectic nature of the uniform limits of sequences of symplectic diffeomorphisms isotopic to the identity map: This provides various symplectic analogues of a result that was proved by Hofer–Zehnder [10] (for compactly supported Hamiltonian diffeomorphisms on $\mathbb{R}^{2n}$); and reformulated by Oh–Müller [13] for Hamiltonian diffeomorphisms in general. Moreover, we show that Polterovich's regularization method for Hamiltonian paths extends over the whole group of symplectic paths (no matter the choice of the above section), and use this to prove the equality between the two versions of the corresponding Hofer-like metrics defined on the group of time-one maps of all symplectic isotopies: The symplectic analogues of the uniqueness result of Hofer's geometry [14]. This includes some condition(s) that make the latter uniqueness result continues holds in the context of Bus–Leclercq [7], and a short proof of the non-degeneracy of various Hofer-like metrics. Finally, an alternate proof (without appealing to Floer's theory) of a result from flux geometry found by McDuff–Salamon [12] is given, and various symplectic analogues of some approximation results found by Oh–Müller [13] are provided.

1. Introduction

The Hofer geometry originated with the remarkable paper of Hofer that introduced the Hofer topologies on the group of Hamiltonian diffeomorphisms of a symplectic manifold (so-called Hofer metrics, [9]). In particular, Hofer–Zehnder [10] had elaborated almost all the basic formulas and perspectives for the subsequent development of Hamiltonian dynamics based on Hofer's metrics.

E-mail address: tchuiagas@gmail.com.

https://doi.org/10.1016/j.difgeo.2018.09.003

Recently, Banyaga [2] and Bus–Leclercq [7] showed that on a closed symplectic manifold (M, ω), each Hofer's metric generalizes over the whole group of time-one maps of all symplectic isotopies (so-called Hofer-like metrics, [2,7]). Here, generalization means: If the first de Rham cohomology group of M is trivial, then these Hofer-like metrics reduce to some metrics defined on the group of all Hamiltonian diffeomorphisms.

Furthermore, if the above de Rham group is non-trivial, then it is known that the restriction of the $L^{(1,\infty)}$-version of Banyaga's Hofer-like norm to the group of all Hamiltonian diffeomorphisms is equivalent to the usual Hofer norm (see [7] and [16]).

However, we have some thorough discussions based on the usual Hofer's metrics whose symplectic analogues are still unknown (see [10], [11], [13], and [14]). These facts seem to attest that to better understand the geometries behind the Hofer-like metrics (e.g. the study of geodesics in the group of all time-one maps of symplectic isotopies with respect to Hofer-like metrics), further investigations still need to be done. This is one of the main objectives of the present paper: Contribute to the foundation of various Hofer-like geometries.

We organize this paper as follows. In Section 2, we study some relationships between symplectic isotopies and the splitting of the space of closed 1-forms $\mathcal{Z}^1(M)$ as a direct sum

$$\mathcal{Z}^1(M) = \mathbb{H}^1(M, \mathcal{S}) \oplus_{\mathcal{S}} \mathbb{B}^1(M), \tag{1.1}$$

with respect to an arbitrary linear section $\mathcal{S}$ of the natural projection $\pi : \mathcal{Z}^1(M) \to H^1(M, \mathbb{R})$, where $\mathbb{H}^1(M, \mathcal{S})$ is isomorphic to the first de Rham group $H^1(M, \mathbb{R})$, and $\mathbb{B}^1(M)$ is contained in $\ker \pi$: Subsection 2.4 deals with the $\mathcal{S}$-decomposition of symplectic isotopies which generalizes the usual Hodge decomposition of symplectic isotopies; Subsection 2.5 illustrates some implications of Riemannian geometry in the study of various Hofer-like geometry, while in Subsection 2.10, we use the splitting in (1.1) to prove that Polterovich's regularization method for Hamiltonian isotopies admits various symplectic analogues (no matter the choice of the linear section $\mathcal{S}$): This general regularization process could play a paramount role in the study of geodesics with respect to various Hofer-like metrics. As a consequence of Proposition 2.4, we give an alternative proof (without appealing to Floer's theory) of a result from flux geometry (Proposition 2.7) found by McDuff–Salamon [12]. Section 3 contains the main results of this paper: The first main result Theorem 3.3 shows in particular that for each section $\mathcal{S}$, one can use the L^∞-Hofer-like lengths to investigate the symplectic nature of the C^0-limit of sequences of symplectic diffeomorphisms isotopic to the identity map (without appealing to the positivity result of any symplectic displacement energy). Here, we exhibit the existence of a larger number of non-Hamiltonian symplectic isotopies whose Hofer-like lengths are symmetric (Example 3.1).

The second main result Theorem 3.8 shows that the Hofer-like geometry resulting from each of the above splitting is independent to the choice of a corresponding version (L^∞-version or $L^{(1,\infty)}$-version) of the Hofer-like metrics. This includes some condition(s) that make the latter uniqueness result holds in the context of Bus–Leclercq [7], and a short proof of the non-degeneracy of these Hofer-like metrics. The last Section deals with various symplectic analogues of some approximation lemmas found by Oh–Müller [13].

2. Preliminaries

Let M be a smooth closed manifold of dimension $2n$. In brief, a 2-form ω on M is called a symplectic form if it is closed and non-degenerate. Then, a symplectic manifold is a manifold which can be equipped with a symplectic form. In particular, note that any symplectic manifold is oriented, and not all even dimensional manifolds can be equipped with a symplectic form.

In the rest of this paper, we shall always assume that M is a closed manifold that admits a symplectic form ω, and we shall fix a Riemannian metric g on M (any differentiable manifold M can be equipped