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The cross-section of a spherical double cone

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We show that the poset of $SL(n)$-orbit closures in the product of two partial flag varieties is a lattice if the action of $SL(n)$ is spherical.

1. Introduction

Let G be a connected reductive algebraic group. A normal algebraic variety X is called a spherical G-variety if there exists an algebraic action $G \times X \to X$ such that the restriction of the action to a Borel subgroup B of G has an open orbit in X. In this case, we say that the action is spherical.

Let $P_1, \dots, P_k \subset G$ be a list of parabolic subgroups containing the same Borel subgroup B and let X denote the product variety $X = G/P_1 \times \cdots \times G/P_k$. Then X is a smooth, hence normal, G-variety via the diagonal action. The study of functions on an affine cone over X is important for understanding the decompositions of tensor products

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of representations of G, see [15,10]. In particular, determining when the diagonal action of G on X is spherical is important for understanding the multiplicity-free representations of G. In his ground breaking article [6], Littelmann initiated the classification problem and gave a list of all possible pairs of maximal parabolic subgroups P_1, P_2 such that $G/P_1 \times G/P_2$ is a spherical G-variety. In [7], for group $G = SL(n)$ and in [8] for $G = Sp(2n)$, Magyar, Weyman, and Zelevinski classified the parabolic subgroups $P_1, \dots, P_k$ such that the product $X = G/P_1 \times \cdots \times G/P_k$ is a spherical G-variety. According to [7], if X is a spherical G-variety, then the number of factors is at most 3, and $k = 3$ occurs in only special cases. Therefore, the gist of the problem lies in the case $k = 2$. This case is settled in full detail by Stembridge. In [12], for a semisimple complex algebraic group G, Stembridge listed all pairs of parabolic subgroups (P_1, P_2) such that $G/P_1 \times G/P_2$ is a spherical G-variety.

For motivational purposes, we will mention some recent related developments. Let K be a connected reductive subgroup of G and let P be a parabolic subgroup of G. One of the major open problems in the classification of spherical actions is the following: What are the possible triplets (G, K, P) such that G/P is a spherical K-variety? When K is a Levi subgroup of a parabolic subgroup Q, this question is equivalent to asking when $G/Q \times G/P$ is a spherical G-variety via diagonal action; it has a known solution as we mentioned earlier. For an explanation of this equivalence, see [1, Lemma 5.4]. In [1], Avdeev and Petukhov gave a complete answer to the above question in the case $G = SL(n)$. If we assume that K is a symmetric subgroup of G, then our initial question is equivalent to asking when $G/P \times K/B_K$ has an open K-orbit via its diagonal action. Here, B_K is a Borel subgroup of K. In this case, the answer is recorded in [4]. See also the related work of Pruijssen [14]. Finally, let us mention another extreme situation where the answer is known: G is an exceptional simple group, P is a maximal parabolic subgroup, and K is a maximal reductive subgroup of G, see [9].

We go back to the products of flag varieties and let P and Q be two parabolic subgroups from G. From now on we will call a product variety of the form $G/P \times G/Q$ a double flag variety. If the diagonal action of G on a double flag variety $X = G/P \times G/Q$ is spherical, then we will call X a spherical double flag variety for G. As it is shown by Littelmann in his previously mentioned article, the problem of deciding if a double flag variety is spherical or not is closely related to a study of the invariants of a maximal unipotent subgroup in the coordinate ring of an affine cone over X. In turn, this study is closely related to the combinatorics of the G-orbits in X. In this regard, our goal in this note is to prove the following result on the poset of inclusion relationships between the G-orbit closures in a spherical double flag variety.

Theorem 1.1. *Let G denote the special linear group $SL(n+1)$. If X is a spherical double flag variety for G, then the poset of G-orbit closures in X is a lattice.*