

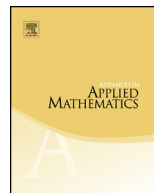


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# The cross-section of a spherical double cone



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## ABSTRACT

We show that the poset of  $SL(n)$ -orbit closures in the product of two partial flag varieties is a lattice if the action of  $SL(n)$  is spherical.

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## 1. Introduction

Let  $G$  be a connected reductive algebraic group. A normal algebraic variety  $X$  is called a spherical  $G$ -variety if there exists an algebraic action  $G \times X \rightarrow X$  such that the restriction of the action to a Borel subgroup  $B$  of  $G$  has an open orbit in  $X$ . In this case, we say that the action is spherical.

Let  $P_1, \dots, P_k \subset G$  be a list of parabolic subgroups containing the same Borel subgroup  $B$  and let  $X$  denote the product variety  $X = G/P_1 \times \dots \times G/P_k$ . Then  $X$  is a smooth, hence normal,  $G$ -variety via the diagonal action. The study of functions on an affine cone over  $X$  is important for understanding the decompositions of tensor products

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of representations of  $G$ , see [15,10]. In particular, determining when the diagonal action of  $G$  on  $X$  is spherical is important for understanding the multiplicity-free representations of  $G$ . In his ground breaking article [6], Littelmann initiated the classification problem and gave a list of all possible pairs of maximal parabolic subgroups  $P_1, P_2$  such that  $G/P_1 \times G/P_2$  is a spherical  $G$ -variety. In [7], for group  $G = SL(n)$  and in [8] for  $G = Sp(2n)$ , Magyar, Weyman, and Zelevinski classified the parabolic subgroups  $P_1, \dots, P_k$  such that the product  $X = G/P_1 \times \dots \times G/P_k$  is a spherical  $G$ -variety. According to [7], if  $X$  is a spherical  $G$ -variety, then the number of factors is at most 3, and  $k = 3$  occurs in only special cases. Therefore, the gist of the problem lies in the case  $k = 2$ . This case is settled in full detail by Stembridge. In [12], for a semisimple complex algebraic group  $G$ , Stembridge listed all pairs of parabolic subgroups  $(P_1, P_2)$  such that  $G/P_1 \times G/P_2$  is a spherical  $G$ -variety.

For motivational purposes, we will mention some recent related developments. Let  $K$  be a connected reductive subgroup of  $G$  and let  $P$  be a parabolic subgroup of  $G$ . One of the major open problems in the classification of spherical actions is the following: What are the possible triplets  $(G, K, P)$  such that  $G/P$  is a spherical  $K$ -variety? When  $K$  is a Levi subgroup of a parabolic subgroup  $Q$ , this question is equivalent to asking when  $G/Q \times G/P$  is a spherical  $G$ -variety via diagonal action; it has a known solution as we mentioned earlier. For an explanation of this equivalence, see [1, Lemma 5.4]. In [1], Avdeev and Petukhov gave a complete answer to the above question in the case  $G = SL(n)$ . If we assume that  $K$  is a symmetric subgroup of  $G$ , then our initial question is equivalent to asking when  $G/P \times K/B_K$  has an open  $K$ -orbit via its diagonal action. Here,  $B_K$  is a Borel subgroup of  $K$ . In this case, the answer is recorded in [4]. See also the related work of Pruijssen [14]. Finally, let us mention another extreme situation where the answer is known:  $G$  is an exceptional simple group,  $P$  is a maximal parabolic subgroup, and  $K$  is a maximal reductive subgroup of  $G$ , see [9].

We go back to the products of flag varieties and let  $P$  and  $Q$  be two parabolic subgroups from  $G$ . From now on we will call a product variety of the form  $G/P \times G/Q$  a double flag variety. If the diagonal action of  $G$  on a double flag variety  $X = G/P \times G/Q$  is spherical, then we will call  $X$  a spherical double flag variety for  $G$ . As it is shown by Littelmann in his previously mentioned article, the problem of deciding if a double flag variety is spherical or not is closely related to a study of the invariants of a maximal unipotent subgroup in the coordinate ring of an affine cone over  $X$ . In turn, this study is closely related to the combinatorics of the  $G$ -orbits in  $X$ . In this regard, our goal in this note is to prove the following result on the poset of inclusion relationships between the  $G$ -orbit closures in a spherical double flag variety.

**Theorem 1.1.** *Let  $G$  denote the special linear group  $SL(n+1)$ . If  $X$  is a spherical double flag variety for  $G$ , then the poset of  $G$ -orbit closures in  $X$  is a lattice.*

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