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On contact graphs of totally separable packings in low dimensions

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ABSTRACT

The *contact graph* of a packing of translates of a convex body in Euclidean d-space $\mathbb{E}^d$ is the simple graph whose vertices are the members of the packing, and whose two vertices are connected by an edge if the two members touch each other. A packing of translates of a convex body is called *totally separable*, if any two members can be separated by a hyperplane in $\mathbb{E}^d$ disjoint from the interior of every packing element.

We give upper bounds on the maximum degree (called *separable Hadwiger number*) and the maximum number of edges (called *separable contact number*) of the contact graph of a totally separable packing of n translates of an arbitrary smooth convex body in $\mathbb{E}^d$ with $d = 2, 3, 4$. In the proofs, linear algebraic and convexity methods are combined with volumetric and packing density estimates based on the underlying isoperimetric (resp., reverse isoperimetric) inequality.

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1. Introduction

We denote the d-dimensional Euclidean space by $\mathbb{E}^d$, and the unit ball centered at the origin $\mathbf{o}$ by $\mathbf{B}^d$. A *convex body* $\mathbf{K}$ is a compact convex subset of $\mathbb{E}^d$ with nonempty interior. Throughout the paper, $\mathbf{K}$ always denotes a convex body in $\mathbb{E}^d$. If $\mathbf{K} = -\mathbf{K} := \{-x : x \in \mathbf{K}\}$, then $\mathbf{K}$ is said to be $\mathbf{o}$-*symmetric*. $\mathbf{K}$ is said to be *smooth* if at every point on the boundary $\operatorname{bd}\mathbf{K}$ of $\mathbf{K}$, the body $\mathbf{K}$ is supported by a unique hyperplane of $\mathbb{E}^d$. $\mathbf{K}$ is *strictly convex* if the boundary of $\mathbf{K}$ contains no nontrivial line segment.

The *kissing number problem* asks for the maximum number $k(d)$ of non-overlapping translates of $\mathbf{B}^d$ that can touch $\mathbf{B}^d$. Clearly, $k(2) = 6$. To date, the only known kissing number values are $k(3) = 12$ [20], $k(4) = 24$ [16], $k(8) = 240$ [17], and $k(24) = 196560$ [17]. For a survey of kissing numbers we refer the interested reader to [7].

Generalizing the kissing number, the *Hadwiger number* or *the translative kissing number* $H(\mathbf{K})$ of a convex body $\mathbf{K}$ is the maximum number of non-overlapping translates of $\mathbf{K}$ that all touch $\mathbf{K}$. Given the difficulty of the kissing number problem, determining Hadwiger numbers is highly nontrivial with few exact values known for $d \geq 3$. The best general upper and lower bounds on $H(\mathbf{K})$ are due to Hadwiger [12] and Talata [22] respectively, and can be expressed as

$$2^{cd} \leq H(\mathbf{K}) \leq 3^d - 1, \tag{1}$$

where c is an absolute constant and equality holds in the right inequality if and only if $\mathbf{K}$ is an affine d-dimensional cube [11].

A packing of translates of a *convex domain*, that is, a planar convex body, $\mathbf{K}$ in $\mathbb{E}^2$ is said to be *totally separable* if any two packing elements can be separated by a line of $\mathbb{E}^2$ disjoint from the interior of every packing element. This notion was introduced by G. Fejes Tóth and L. Fejes Tóth [10].

We can define a totally separable packing of translates of a d-dimensional convex body $\mathbf{K}$ in a similar way by requiring any two packing elements to be separated by a hyperplane in $\mathbb{E}^d$ disjoint from the interior of every packing element [6,13].

Recall that the *contact graph* of a packing of translates of $\mathbf{K}$ is the simple graph whose vertices are the members of the packing, and whose two vertices are connected by an edge if and only if the two members touch each other. In this paper we investigate the maximum degree (called *separable Hadwiger number*), as well as the maximum number of edges (called the *maximum separable contact number*) of the contact graphs of totally separable packings by a given number of translates of a smooth or strictly convex body $\mathbf{K}$ in $\mathbb{E}^d$. This extends and generalizes the results of [4] and [6]. The details follow.

1.1. Separable Hadwiger numbers

It is natural to introduce the totally separable analogue of the Hadwiger number as follows [4].