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New insights on marginally trapped surfaces: The hedgehog theory point of view

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In this paper, we will try to argue and to show through fundamental examples that (a very huge class of) marginally trapped surfaces arise naturally from a 'lightlike co-contact structure', exactly in the same way as Legendrian fronts arise from a contact one (by projection of a Legendrian submanifold to the base of a Legendrian fibration), and that there is an adjunction relationship between both notions. We especially focus our interest on marginally trapped hedgehogs and study their relationships with Laguerre geometry and Brunn–Minkowski theory.

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1. Introduction and statement of main results

Trapped surfaces were introduced in general relativity by R. Penrose [12] to study singularities of spacetimes. They appeared in a natural way earlier in the work of Blaschke, in the context of conformal and Laguerre geometry [1]. These surfaces play an extremely important role in general relativity where they are of central importance in the study of black holes, those regions of spacetime where everything is trapped, and nothing can escape, even light. A closed embedded spacelike 2-surface of a 4-dimensional spacetime is said to be trapped if its mean curvature vector is everywhere timelike. The limiting case of marginally trapped surfaces, i.e. surfaces whose mean curvature vector is everywhere lightlike, play the role of apparent horizons of black holes. Mathematically, marginally trapped surfaces are regarded as spacetime analogues of minimal surfaces in Riemannian geometry. Even though they received considerable attention both from mathematicians and physicists, these surfaces are still not very well understood. For a recent survey on marginally trapped surfaces, see the book by B.Y. Chen [3].

What we will try to argue in this paper, and to show through fundamental examples, is that:

(A very huge class of) marginally trapped surfaces arise naturally from a 'lightlike co-contact structure', exactly in the same way as Legendrian fronts arise from a contact one (by projection of a Legendrian submanifold to the base of a Legendrian fibration), and there is an adjunction relationship between both notions.

In addition, a huge class of marginally trapped surfaces correspond by adjunction to hedgehogs (envelopes parametrized by their Gauss map) and can thus benefit directly from contributions of hedgehog theory, which can be seen as an extension of the Brunn–Minkowski one (see e.g. [10]). This correspondence is naturally promising in terms of new geometric inequalities, and we know how important geometric inequalities are in gravitation. We will give examples of geometric inequalities involving hedgehogs and marginally trapped surfaces in Subsection 1.2.

In order to explain precisely what we mean here by a '*lightlike co-contact structure*', let us begin by the presentation of a fundamental example in the 4-dimensional Lorentz–Minkowski space $\mathbb{L}^4$. This example will be detailed in Subsection 1.3.

1.1. Marginally trapped hedgehogs or co-hedgehogs in $\mathbb{L}^4$

1.1.1. Characterization and definitions in $\mathbb{L}^4$

For simplicity, we will restrict our presentation to surfaces in $\mathbb{L}^4$ but our results extend, without much change, to higher dimensions. To any $h \in C^\infty\left(\mathbb{S}^2;\mathbb{R}\right)$ corresponds the envelope $\mathcal{H}_h$ of the family $\left(P_h\left(u\right)\right)_{u\in\mathbb{S}^2}$ of cooriented planes of $\mathbb{R}^3$ with equation (E) $\langle x,u\rangle = h\left(u\right)$, where $\langle .,.\rangle$ is the standard scalar product on $\mathbb{R}^3$. We say that $\mathcal{H}_h$ is the hedgehog with support function h. From (E) and the contact condition $\langle dx,u\rangle = 0$, we deduce