ARTICLE IN PRESS

Journal of Computational and Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# The analytical solution and numerical solutions for a two-dimensional multi-term time fractional diffusion and diffusion-wave equation

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## HIGHLIGHTS

- A new two-dimensional multi-term time-fractional diffusion and diffusion.
- Wave equation (2D-MT-TFD-DWE) is considered.
- The analytical solution for the 2D-MT-TFD-DWE is derived.
- A novel implicit difference method (IDM) for 2D-MT-TFD-DWE is proposed.
- The stability and convergence of the IDM are proved by the energy method.
- Numerical examples are given.

## ARTICLE INFO

*Article history:*
Received 14 August 2017
Received in revised form 2 January 2018

*MSC:*
65M20
65L06
65R10
26A33

*Keywords:*
Multi-term time-fractional diffusion equation
Numerical method
Stability
Convergence
Method of separation of variables
Energy method

## ABSTRACT

In this paper we consider the analytical and numerical solutions for a two-dimensional multi-term time-fractional diffusion and diffusion-wave equation. We derive the analytical solution for the equation using the method of separation of variables and properties of the multivariate Mittag-Leffler function. An implicit difference approximation is constructed. Stability and convergence analysis of the numerical scheme are proved by the energy method. Numerical examples are constructed to evaluate the working of the numerical scheme as compared to theoretical analysis.



## 1. Introduction

Fractional diffusion equations have been used to describe important physical phenomena that arise in the natural science and engineering fields [1–4]. Many works have been devoted to the one-dimensional [5–8] or single-term time and/or space

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https://doi.org/10.1016/j.cam.2018.05.020

fractional diffusion equations [9–11]. On the other hand, the multi-term time-fractional operator has been found useful in describing complex physical and physiological systems [12–14]. When the orders of the time-fractional derivatives belong to the interval (0, 1), the equations are called multi-term time-fractional diffusion equations. When the fractional orders belong to the interval (1, 2), the equations are called multi-term time-fractional wave equations. When the fractional orders belong to the interval (0, 2), they are known as multi-term time-fractional diffusion-wave equations. Physical backgrounds, numerical analysis and techniques are different for the three cases. Jiang et al. [15] proposed some analytical techniques to solve three types of multi-term time-space Caputo–Riesz fractional advection–diffusion equations with nonhomogeneous Dirichlet boundary conditions. Liu et al. [16] found numerical solutions for the multi-term time-fractional equations in one dimension with orders of the time-fractional derivatives belonging to the intervals [0, 1), [1, 2), [0, 2) and also [0,3), [2,3) and [2,4), separately. The fractional predictor–corrector method was used for approximations in the latter three cases. Bhrawy and Zaky [17] proposed and analyzed an efficient operational formulation of the spectral tau method for multi-term time-space fractional differential equations with Dirichlet boundary conditions. Zhao et al. [18] derived the analytical solution by use of a method of separating variables and established a fully-discrete approximate scheme by means of nonconforming finite element method and $L_1$-approximation for a two-dimensional multi-term time-fractional sub-diffusion equation. Reutskiy [19] used a combined method of separating variables and Fourier expansion with backward substitution to solve a wide class of fractional partial differential equations including diffusion-wave equations, modified anomalous fractional sub-diffusion equations and time-fractional telegraph equations. Solutions of the multi-term fractional differential equations in higher dimensions are still under development.

The time-fractional diffusion and diffusion-wave equation is a generalization of the telegraph equation. During the past decade, many authors have investigated this subject. Orsingher and Zhao [20] analyzed space-fractional telegraph equation and obtained the Fourier transform of its fundamental solution. Beghin and Orsingher [21] proved that the fundamental solution to the Cauchy problem for the fractional telegraph equation with partial fractional derivatives of rational order can be expressed as the distribution of the composition of two processes. And they obtained explicit expressions for the probability distribution of a telegraph process with a random time and for the characteristic function of a telegraph process stopped at stable-distributed times. Anh and Leonenko [22] presented the Green functions and spectral representations of the mean-square solutions of the fractional diffusion-wave equations with random initial conditions. Orsingher and Beghin [23] studied the fundamental solutions to time-fractional telegraph equations of order $2\alpha$. And they obtained the Fourier transform of the solutions for any $\alpha$ to give a representation of their inverse, in terms of stable densities. Kolesnik and Pinsky [24] resolved the long-standing problem of describing the multidimensional random evolutions by means of the telegraph equations and showed that the multidimensional random evolutions are driven by the hyperparabolic operators composed of the telegraph operators and their integer powers. Meerschaert et al. [25] showed that the fractional wave equation governs a stochastic model for wave propagation, with deterministic time replaced by the inverse of a stable subordinator whose index is one-half the order of the fractional time derivative.

The present investigation focuses on the following two-dimensional multi-term time-fractional diffusion and diffusion-wave equation. This model is a generalization of the so-called Relativistic Brownian motion [26].

$$\begin{aligned} &P_{\beta_1\beta_2\cdots\beta_{k_2}}\left({}_0^CD_t\right)u(x,y,t)+d_1\frac{\partial u(x,y,t)}{\partial t}+P_{\alpha_1\alpha_2\cdots\alpha_{k_1}}\left({}_0^CD_t\right)u(x,y,t)\\ &+d_2u(x,y,t)=d_3\Delta u(x,y,t)+f(x,y,t),\\ &\quad (x,y)\in\Omega=(0,L_x)\times(0,L_y),\quad 0<t\le T, \end{aligned} \tag{1}$$

with the initial conditions:

$$u(x,y,0)=\phi(x,y),\qquad \frac{\partial u(x,y,0)}{\partial t}=\psi(x,y),\qquad (x,y)\in\Omega, \tag{2}$$

and the homogeneous boundary conditions:

$$\begin{aligned} &u(0,y,t)=u(L_x,y,t)=0,\quad 0\le y\le L_y,\quad 0\le t\le T,\\ &u(x,0,t)=u(x,L_y,t)=0,\quad 0\le x\le L_x,\quad 0\le t\le T, \end{aligned} \tag{3}$$

where

$$P_{\alpha_1\alpha_2\cdots\alpha_{k_1}}\left({}_0^CD_t\right)u(x,y,t)=\sum_{i_1=1}^{k_1}p_{i_1}\cdot\left({}_0^CD_t^{\alpha_{i_1}}u(x,y,t)\right), \tag{4}$$

$$P_{\beta_1\beta_2\cdots\beta_{k_2}}\left({}_0^CD_t\right)u(x,y,t)=\sum_{i_2=1}^{k_2}q_{i_2}\cdot\left({}_0^CD_t^{\beta_{i_2}}u(x,y,t)\right), \tag{5}$$

$0<\alpha_1<\alpha_2<\cdots<\alpha_{k_1}<1$, $1<\beta_1<\beta_2<\cdots<\beta_{k_2}<2$, $d_i>0\,(i=1,2,3)$, $p_{i_1},q_{i_2}\in\mathbb{R}^+$, $i_1,i_2,k_1,k_2\in\mathbb{N}$, and $\phi(x,y)$ and $\psi(x,y)$ are continuous functions. In the above expressions, $\Delta$ is the two-dimensional Laplacian: $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$,

Please cite this article in press as: S. Shen, et al., The analytical solution and numerical solutions for a two-dimensional multi-term time fractional diffusion and diffusion-wave equation, Journal of Computational and Applied Mathematics (2018), https://doi.org/10.1016/j.cam.2018.05.020.