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Combinatorics of n-color cyclic compositions

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ABSTRACT

Integer compositions and related enumeration problems have been of interest to combinatorialists and number theorists for a long time. The cyclic and colored analogues of this concept, although interesting, have not been extensively studied. In this paper we explore the combinatorics of n-color cyclic compositions, presenting generating functions, bijections, asymptotic formulas related to the number of such compositions, the number of parts, and the number of restricted parts.

1. Introduction

The interest in integer compositions dates back to 1893 [7, p. 1]. As is well known, a *composition* of a given positive integer ℓ is a sequence of positive integers $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ such that $\sum_{i=1}^{k} \sigma_i = \ell$. Each σ_i is called a *part* in the composition. For example, there are four compositions of 3:

$$3, \qquad 1+2, \qquad 2+1, \qquad 1+1+1.$$

A colored analogue of compositions, called the *n-color compositions*, seems to be first introduced in [1]. In an n-color composition each part of size n has n possible colors. For instance, in the composition $1+2$ of 3, the part 2 has two possible colors which we label as 2_1 and 2_2. Hence, there are eight n-color compositions of 3:

$$3_1, \quad 3_2, \quad 3_3, \quad 1_1+2_1, \quad 1_1+2_2, \quad 2_1+1_1, \quad 2_2+1_1, \quad 1_1+1_1+1_1.$$

Following [1], a series of studies has been presented on the n-color compositions including various bijections which establish the combinatorial connections between n-color compositions and other objects. For some examples of these studies one may see [2,4,8,12–14]. In particular, a nice survey is provided in [3], on the combinatorics of n-color compositions.

Cyclic compositions, first considered in [15] and enumerated via generating functions in [5], can be considered as a partition of the set of compositions under the equivalence relation T, where T is any cyclic shift of the parts of a composition. For example, Fig. 1 shows a cyclic composition of 10 corresponding to the equivalence class $2+2+1+2+2+1 = 2+1+2+2+1+2 = 1+2+2+1+2+2 = 10$. As one would imagine, the enumeration of various objects in cyclic compositions dramatically differs from that in compositions. Some interesting properties of cyclic compositions were discussed in [10].

Much of the study on compositions focuses on the enumeration of compositions, parts, or sub-word patterns under various restrictions. A nice summary of known results can be found in [7] and the references therein. In this paper, we

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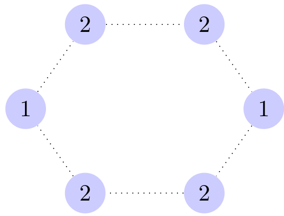

Fig. 1. A cyclic composition of 10.

explore such questions for n-color cyclic compositions. When considered as n-color (non-cyclic) compositions, $1_1 + 2_1$ and $2_1 + 1_1$ are distinct, whereas they would be the same if considered as n-color cyclic compositions since one is a cyclic shift of the other. Thus, the six n-color cyclic compositions of 3 are

$$3_1, \qquad 3_2, \qquad 3_3, \qquad 1_1 + 2_1, \qquad 1_1 + 2_2, \qquad 1_1 + 1_1 + 1_1.$$

As a slightly less trivial example, we have that

$$1_1 + 2_1 + 4_3 + 1_1 + 2_1 + 4_3, \qquad 2_1 + 4_3 + 1_1 + 2_1 + 4_3 + 1_1, \qquad 4_3 + 1_1 + 2_1 + 4_3 + 1_1 + 2_1$$

are all considered the same as n-color cyclic compositions of 14. The results in this paper are in some sense comparable to [14], where generating functions for six new restricted n-color compositions have been found and the associated combinatorics have been discussed. Let $\mathscr{NCC}_\ell$ be the set of n-color cyclic compositions of ℓ. For a set S of compositions, we use $n(S)$, $np(S)$, $n(S; \mathscr{P})$, and $np(S; \mathscr{P})$ to denote, respectively, the number of compositions, the total number of parts in all compositions in S, the number of compositions with all parts satisfying some condition $\mathscr{P}$, and the total number of parts satisfying some condition $\mathscr{P}$ in all compositions in S. Similarly, we use $N[S](x)$, $NP[S](x)$, $N[S; \mathscr{P}](x)$, and $NP[S; \mathscr{P}](x)$ to denote the corresponding ordinary generating functions. For instance, $n(\mathscr{NCC}_\ell)$ is the number of n-color cyclic compositions of ℓ and

$$N[\mathscr{NCC}_\ell](x) = \sum_{\ell \geq 0} n(\mathscr{NCC}_\ell) \cdot x^\ell.$$

Following the analytical tools developed in [5], we easily obtain the generating functions (whose proofs we postpone to Section 4)

$$N[\mathscr{NCC}_\ell](x) = \sum_{s \geq 1} \frac{\varphi(s)}{s} \log\left(\frac{(1-x^s)^2}{(1-3x^s+x^{2s})}\right) \tag{1.1}$$

and

$$NP[\mathscr{NCC}_\ell](x) = \sum_{s \geq 1} \varphi(s) \left(\frac{x^s}{1-3x^s+x^{2s}}\right) \tag{1.2}$$

where $\varphi(s)$ is the Euler function. As an application of these generating functions, from (1.1) and (1.2) we have

$$n(\mathscr{NCC}_\ell) \sim \frac{\tau^\ell}{\ell}$$

and

$$np(\mathscr{NCC}_\ell) \sim \frac{\tau^\ell}{\sqrt{5}}$$

where $\tau = \frac{2}{3-\sqrt{5}}$. These formulas immediately imply the following.

Corollary 1.1. *The average number of parts in compositions of* $\mathscr{NCC}_\ell$ *is*

$$\frac{\ell}{\sqrt{5}}$$

as $\ell \to \infty$.

Thus, the average number of parts goes to infinity as ℓ goes to infinity. Intuitively, this is saying that the number of parts which are small in size (compared to ℓ) vastly outnumbers the number of larger parts. Similarly, letting $\mathscr{P}_1 \leftrightarrow (\equiv i \pmod{m})$