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New sufficient conditions for bipancyclicity of balanced bipartite digraphs

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ABSTRACT

The main results provide sufficient conditions for balanced bipartite digraphs to be bipancyclic. These are analogues to well-known results on pancyclic digraphs.

1. Introduction

We use standard notation and terminology, cf. [3]. Let $D = (V, A)$ be a digraph with the vertex set V and the arc set A. If (u, v) is an arc in D, u is called a *tail* and v a *head*. A *degree* of a vertex v in D is the number of arcs incident to v and is denoted $d_D(v)$. Symbols $d_D^+(v)$ and $d_D^-(v)$ denote *out-degree* and *in-degree* of a vertex v in D which are the numbers of arcs with tail and head in v, respectively.

Two vertices u and v are *adjacent* if $(u, v) \in A$ or $(v, u) \in A$. A digraph D is called a *tournament* if every two distinct vertices u and v are adjacent by a single arc, i.e., either $(u, v) \in A$ or $(v, u) \in A$. A *directed path* of length k in D is a sequence $P = \langle v_1, v_2, \ldots, v_{k+1}\rangle$ of pairwise distinct vertices in D such that, for each $1 \leq i \leq k$, (v_i, v_{i+1}) is an arc in D. If an arc (v_{k+1}, v_1) is attached to P then we get a *directed cycle* of length $k + 1$. A vertex v is *reachable* from a vertex u in D if there exists a directed path from u to v. In particular, v is reachable from itself. A digraph D is *strongly connected* if, for every two vertices u and v, v is reachable from u and u is reachable from v. A digraph D of order n is *hamiltonian* if it has a directed cycle of length n. D is *pancyclic* if it contains directed cycles of all lengths $2, 3, \ldots, n$.

A *bipartite digraph* $D = (X, Y; A)$ has the vertex set partitioned into two *partite sets* X and Y of cardinalities a and b, respectively, where A denotes the set of arcs; each arc has one vertex in X and the other in Y. If $a = b$ then D is called *balanced*. $K^*_{a,b}$ denotes a *complete bipartite digraph* with partite sets of cardinalities a and b. A *matching M from X to Y* is a set of arcs such that any vertex in $X \cup Y$ is incident with at most one arc in A and moreover each arc in M has its tail in X and a head in Y; M is *perfect* if each vertex has incident arc in M. A balanced bipartite digraph D of order $2a$ is *bipancyclic* if it contains directed cycles of all even lengths $2, 4, \ldots, 2a$.

Thomassen [7] proved an Ore-type sufficient condition for pancyclicity of digraphs.

Theorem 1 ([7]). *Let D be a strongly connected digraph of order $n \geq 3$ such that $d_D(u) + d_D(v) \geq 2n$ whenever u and v are nonadjacent. Then either D is pancyclic, or D is a tournament, or else n is even and D is isomorphic to $K^*_{\frac{n}{2},\frac{n}{2}}$.*

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An analogous assertion holds if a digraph satisfies the Woodall's condition.

Corollary 2 ([7]). *Let D be a digraph of order $n \geq 3$. If $d_D^+(u) + d_D^-(v) \geq n$ for each pair of vertices u and v such that there is no arc from u to v, then either D is pancyclic or else n is even and D is isomorphic to $K^*_{\frac{n}{2},\frac{n}{2}}$.*

Adamus and Adamus [1] proved a sharp Ore-type condition for hamiltonicity in balanced bipartite digraphs.

Theorem 3 ([1]). *Let $D = (X, Y; A)$ be a balanced bipartite digraph such that $|X| = |Y| = a \geq 2$. If $d_D^+(u) + d_D^-(v) \geq a + 2$ for each two vertices u and v from distinct partite sets such that $(u, v) \notin A$, then D is hamiltonian.*

Moreover, Adamus, Adamus and Yeo [2] proved a sharp Meyniel-type condition for hamiltonicity in balanced bipartite digraphs.

Theorem 4 ([2]). *Let $D = (X, Y; A)$ be a balanced bipartite digraph such that $|X| = |Y| = a \geq 2$. Then D is hamiltonian provided one of the following holds:*
(a) $d_D(u) + d_D(v) \geq 3a + 1$ for each pair of nonadjacent vertices $u, v \in X \cup Y$,
(b) D is strongly connected and $d_D(u) + d_D(v) \geq 3a$ for each pair of nonadjacent vertices $u, v \in X \cup Y$.

The main results of the paper improve the above assertions. In particular, the assumptions in Theorem 3 turn out to be sufficient for bipancyclicity.

Theorem 5. *Let $D = (X, Y; A)$ be a balanced bipartite digraph, $|X| = |Y| = a \geq 2$. If $d_D^+(u) + d_D^-(v) \geq a + 2$ for each pair of vertices u and v from distinct partite sets such that $(u, v) \notin A$, then D is bipancyclic.*

Weaker conditions than those in Theorem 4(a) guarantee stronger assertion.

Theorem 6. *Let $D = (X, Y; A)$ be a balanced bipartite digraph, $|X| = |Y| = a \geq 2$, and moreover $d_D(u) + d_D(v) \geq 3a + 1$ for each two vertices u, v either both in X or both in Y. Then D is bipancyclic.*

2. Proofs

A bipartite graph $G = (X, Y; E)$ is called a *half-base* of $D = (X, Y; A)$ if it has the same bipartition (X, Y) of vertices and each edge $\{x, y\} \in E$ is obtained from an arc $(x, y) \in (X, Y)$ in D by ignoring its direction; arcs in (Y, X) are disregard.

Moon and Moser [6] provided well-known sufficient condition for a balanced bipartite graph to be hamiltonian.

Theorem 7 ([6]). *If $G = (X, Y, E)$ is a balanced bipartite graph, where $|X| = |Y| = a$, in which $d_G(x) + d_G(y) > a$ for every pair of nonadjacent vertices $x \in X$ and $y \in Y$, then G is hamiltonian.*

Ferrara, Jacobson and Powell [4] characterized all nonhamiltonian balanced bipartite graphs G such that $\min\{d_G(x) + d_G(y) : \{x, y\} \notin E, x \in X, y \in Y\} = a$. Namely, G is either one of two graphs on 8 vertices or G belongs to a given infinite class of graphs. It is immediate to verify that all these graphs contain perfect matchings.

Corollary 8. *If $G = (X, Y, E)$ is a balanced bipartite graph, $|X| = |Y| = a$, and $d_G(x) + d_G(y) \geq a$ for every pair of nonadjacent vertices $x \in X$ and $y \in Y$, then G contains a perfect matching.* □

An easy consequence of the above results guarantees the existence of perfect matchings in balanced bipartite digraphs under slightly weaker assumptions than those in Theorem 5.

Lemma 9. *Let $D = (X, Y; A)$ be a balanced bipartite digraph, $|X| = |Y| = a$. If $d_D^+(x) + d_D^-(y) \geq a$ for each pair of vertices $x \in X$ and $y \in Y$ such that $(x, y) \notin A$, then D contains a perfect matching from X to Y.*

Proof. Let $G = (X, Y; E)$ be a half-base of $D = (X, Y; A)$. Obviously, $x \in X$ and $y \in Y$ are nonadjacent vertices in G if $(x, y) \notin A$. Therefore $d_G(x) + d_G(y) = d_D^+(x) + d_D^-(y) \geq a$. By Corollary 8, G has a perfect matching M'. Each edge $\{x, y\}$ of M' determines an arc (x, y) of M in D. □

Suppose that $D = (X, Y; A)$ is a balanced bipartite digraph of order $2a$ and moreover D contains a perfect matching M from Y to X which consists of the arcs (y_i, x_i), $i = 1, 2, \ldots, a$. Then, for purpose of the next proofs, we may define a digraph $D^*(M) = (V^*, A^*)$ of order a such that each vertex $v_i \in V^*$ corresponds to a pair $\{x_i, y_i\}$ of vertices in D, $i = 1, 2, \ldots, a$. For each $i \neq j$, $(v_i, v_j) \in A^*$ if and only if $(x_i, y_j) \in A$; when constructing $D^*(M)$ all arcs from Y to X and those of type (x_i, y_i) are ignored. The following property of $D^*(M)$ can be easily observed.

Lemma 10. *If a digraph $D^*(M)$ contains a directed cycle of length k then there is a directed cycle of length 2k in $D = (X, Y; A)$.*