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On the number of Sudoku squares

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ABSTRACT

We provide an upper bound on the number of $n^2 \times n^2$ Sudoku squares, and explain intuitively why there is reason to believe that the bound is tight up to a multiplicative factor of a much smaller order of magnitude. A similar bound is established for Sudoku squares with rectangular regions.

1. Introduction

The Sudoku puzzle is a very popular puzzle, if not the most popular one, in recent years. It (usually) consists of a 9×9 grid, some of whose cells are marked by numbers between 1 and 9. One is required to fill in all the empty cells, again by numbers between 1 and 9, with the following constraints:

- Each row contains each of the numbers $1, 2, \dots, 9$ (each appearing exactly once).
- Each column contains each of the numbers $1, 2, \dots, 9$.
- Each 3×3 mini-grid (see Fig. 1) contains each of the numbers $1, 2, \dots, 9$.

Thus, a (filled) Sudoku square is a special Latin square of order 9. A *Latin square* of order n is an $n \times n$ grid, whose cells are marked by numbers between 1 and n, satisfying the analogues of the first two properties above. Namely, each row and column contains each number exactly once.

Latin squares have been studied extensively from various viewpoints; see, for example, [30]. In particular, there is a lot of literature regarding the number L_n of Latin squares of order n. Euler [5] found this number for $n = 5$ in 1782. Due to the very fast growth of the sequence, our knowledge of L_n for larger sizes of n has not progressed much in the centuries that elapsed. In fact, the largest n for which the value of L_n is currently known exactly is $n = 11$; see [19]. For all values up to 11, see Sequence A002860 in the On-Line Encyclopedia of Integer Sequences [27].

The situation is much better when it comes to estimates of L_n. The following bounds were obtained for L_n (cf. [30, Ch. 17]):

$$n!^{2n}/n^{n^2} \le L_N \le \prod_{k=1}^{n} k!^{n/k}. \tag{1}$$

Note that both bounds have a roughly similar order of magnitude; in fact, employing Stirling's formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot (1 + o(1)),$$

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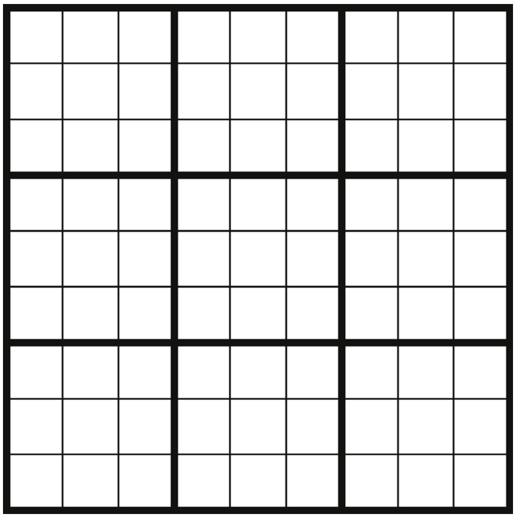

Fig. 1. An empty Sudoku grid.

and routine estimates, we get an effective constant C such that

$$\left(\frac{n}{e^2}\right)^{n^2} e^{n\log n} \le L_n \le \left(\frac{n}{e^2}\right)^{n^2} e^{Cn\log^2 n}. \tag{2}$$

For more information on many aspects of Latin squares (and rectangles) we refer the reader to the survey paper [29].

The mathematics of Sudoku has also been considered from several angles. We refer to [12] for general information on the puzzle, and to [8,25] for discussion of relevant questions of a mathematical nature, and in particular enumerative problems. The question as to the minimum number of clues that must be initially provided, so that the puzzle has a unique solution, has attracted a lot of attention. Many thousands of uniquely solvable 17-clue puzzles have been found [23,24], but only recently has it been verified that there exists no such 16-clue puzzle [18].

As for Latin squares, Sudoku squares admit an immediate generalization to other sizes. An $n^2 \times n^2$ Sudoku grid consists of a grid of this size, in which each row, each column, and each of the $n \times n$ mini-grids composing the whole grid, includes all numbers between 1 and n^2. Our main interest here is in the number S_n of such squares. A further generalization of Sudoku is rectangular Sudoku, in which the mini-grids are not necessarily squares but any rectangles. (However, we emphasize that the whole board is still a square.) Let R, C be positive integers. An $R \times C$ *rectangular Sudoku* is an $RC \times RC$ square, whose cells are marked by numbers between 1 and RC, satisfying the row and column constraints, and divided into RC rectangles of size $R \times C$, in each of which again all the numbers appear exactly once. The case $R = C = n$ is that of classical Sudoku. The case where either $R = 1$ or $C = 1$ reduces to that of a Latin square. We will also be interested in the number $S_{R,C}$ of $R \times C$ rectangular Sudoku squares.

Similarly to Latin squares, the rate of growth of S_n makes it very hard to calculate it exactly. While $S_1 = 1$ and $S_2 = 288$, the next term of the sequence, S_3, is approximately $6.67 \cdot 10^{21}$ (see [7,28]), and is the last term known exactly. In fact, except for S_2, which is trivial to calculate, and $S_{2,3}$, which has been calculated "mathematically" [14], the other known values of $S_{R,C}$ have been calculated employing heavy computations. (For more information we refer to [15].) There are also some heuristics that may be used to estimate S_n, and more generally $S_{R,C}$, that give good approximations in the cases where the exact value is known; see [13] and the references there. There has been even an attempt to estimate S_n by stochastic simulation [22]. Unfortunately, this is a rare-event simulation; unless n is very small, it cannot provide good estimates in reasonable time.

Herzberg and Murty [11] dealt with the question, and were able to bound S_n from above as follows:

$$S_n \le n^{2n^4} e^{-2.5n^4 + O(n^3\log n)}. \tag{3}$$

Clearly, $S_n \le L_{n^2}$. By the right-hand side inequality in (2), this gives:

$$S_n \le n^{2n^4} e^{-2n^4 + O(n^2\log^2 n)}. \tag{4}$$

Thus, Herzberg and Murty's bound of (3) improves the trivial bound of (4) by a factor of roughly $e^{n^4/2}$. (We refer to [26] for a careful derivation of the improved bound.)

Our main result in this paper, stated in Section 2, is an improvement of (3). We also present a lower bound on S_n, that is probably much further from the true value than is our upper bound. A version of the upper bound for rectangular Sudoku is stated also. In Section 3 we discuss some upper bounds for permanents of matrices. These bounds are the main technical