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# Visible lattice points in random walks

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José L. Fernández and Pablo Fernández dedicate this paper to their friend and colleague Javier Cilleruelo, who passed away on May 15, 2016

ABSTRACT

We consider the possible visits to visible points of a random walker moving up and right in the integer lattice (with probability $\alpha$ and $1-\alpha$, respectively), and starting from the origin.

We show that, almost surely, the asymptotic proportion of strings of $k$ consecutive *visible* lattice points visited by such an $\alpha$-random walk is a certain constant $c_k(\alpha)$, which is actually an (explicitly computable) polynomial in $\alpha$ of degree $2\lfloor (k-1)/2 \rfloor$. For $k=1$, this gives that, almost surely, the asymptotic proportion of time the random walker is visible from the origin is $c_1(\alpha)=6/\pi^2$, independently of $\alpha$.



## 1. Introduction

A lattice point $(n, m)$, with $n$ and $m$ integers, is *visible from the origin*, or simply *visible*, if $\gcd(n, m) = 1$. A classical result asserts that the asymptotic proportion of visible lattice points is $6/\pi^2$, when computed over, say, squares centered at the origin of increasing size.

This means, for the first quadrant of the lattice, that if we choose at random a point $(i, j)$, with $i, j$ positive integers, the probability that $(i, j)$ is visible is $6/\pi^2$. That is,

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{\substack{1\le i,j\le N \\ \gcd(i,j)=1}} 1 = \frac{6}{\pi^2} = \prod_{\text{prime } p} \Big(1-\frac{1}{p^2}\Big).$$

This is the big scale point of view of the location of visible points.

The fine structure of the set of visible lattice points has been studied from a variety of view points: its diffraction pattern (see [4] and references therein), its angular distribution [5], its ergodic properties [3], the frequencies of patterns of visible/invisible points [13,3,15], or its percolative properties [14].

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Some of these results point towards the "well-distribution" of this set of visible lattice points; this is to be compared with the fact that there exist big "non-visible regions" (say, squares), meaning that all points within the region are non-visible from the origin. See, for example, the construction in Theorem 5.29 of [1] through the Chinese remainder theorem, and the results about patterns of visible and non-visible lattice points in [13,3,15].

In this paper, we study the following question. We consider a random walker, starting from the origin, and moving up and right in the integer lattice, with probabilities $\alpha$ and $1-\alpha$, respectively. The conclusion of Theorem A is that the asymptotic proportion of time that the random walker remains visible from the origin is $6/\pi^2$, independently of $\alpha$.

This result is not just a consequence of the positive density of the set of visible lattice points. See the discussion of Section 1.1.

More generally, Theorem B establishes the asymptotic proportion of $k$ consecutive visible lattice points visited by the $\alpha$-random walker. Interestingly, this proportion depends on $\alpha$ for $k \geq 3$.

For $\alpha \in (0, 1)$, consider an $\alpha$-random walk in the two-dimensional lattice, starting at $P_0 = (0, 0)$, and given, for $i \geq 1$, by

$$P_i = P_{i-1} + W_i, \quad \text{with} \quad W_i = \begin{cases} (1, 0) \text{ with probability } \alpha, \\ (0, 1) \text{ with probability } 1-\alpha, \end{cases} \tag{1.1}$$

where the jumps $W_i$ are independent. Observe that $W_i = (I_i, 1 - I_i)$, where $I_i$ is a Bernoulli variable with success probability $\alpha$. The walker moves to the neighbor in the lattice one unit up or one unit to the right: only steps $(1, 0)$ and $(0, 1)$ are allowed.

Associated to the $\alpha$-random walk, consider the sequence $(X_i)_{i\geq 1}$ of Bernoulli random variables given by

$$X_i = \begin{cases} 1, & \text{if } P_i \text{ is visible,} \\ 0, & \text{if not,} \end{cases}$$

and write

$$\overline{S}_n = \frac{X_1 + \cdots + X_n}{n}$$

for the variable that registers the proportion of visible points visited by the $\alpha$-random walk in the first $n$ steps. Observe that the variables $X_i$ are not independent.

Our first result reads:

**Theorem A.** *For any $\alpha \in (0, 1)$,*

$$\lim_{n\to\infty} \overline{S}_n = \frac{6}{\pi^2}$$

*almost surely.*

The proof of Theorem A relies on number-theoretical estimates of the mean and the variance of $\overline{S}_n$. See Sections 2 and 3.

Define now, for $k \geq 1$, the random variable

$$\overline{S}_{n,k} = \frac{X_1 \cdots X_k + \cdots + X_n \cdots X_{n+k-1}}{n}$$

that codifies the proportion of $k$ consecutive visible lattice points in the first $n + k - 1$ steps in an $\alpha$-random walk.

**Theorem B.** *For any $\alpha \in (0, 1)$,*

$$\lim_{n\to\infty} \overline{S}_{n,k} = c_k(\alpha) \quad \textit{almost surely,}$$

*where*

$$c_k(\alpha) = b_k(\alpha) \prod_{p \geq k} \Big(1 - \frac{k}{p^2}\Big)$$