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De Bruijn digraphs and affine transformations☆

Aiping Deng[a], Yaokun Wu[a,b]

[a] *Department of Mathematics, Shanghai Jiao Tong University, Shanghai, 200240, China*
[b] *College of Advanced Science and Technology, Dalian University of Technology, Dalian, 116024, China*

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Abstract

Let $\mathbb{Z}_d^n$ be the additive group of $1 \times n$ row vectors over $\mathbb{Z}_d$. For an $n \times n$ matrix T over $\mathbb{Z}_d$ and $\omega \in \mathbb{Z}_d^n$, the affine transformation $F_{T,\omega}$ of $\mathbb{Z}_d^n$ sends x to $xT + \omega$. Let $\langle \alpha \rangle$ be the cyclic group generated by a vector $\alpha \in \mathbb{Z}_d^n$. The affine transformation coset pseudo-digraph $TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ has the set of cosets of $\langle \alpha \rangle$ in $\mathbb{Z}_d^n$ as vertices and there are c arcs from $x + \langle \alpha \rangle$ to $y + \langle \alpha \rangle$ if and only if the number of $z \in x + \langle \alpha \rangle$ such that $F_{T,\omega}(z) \in y + \langle \alpha \rangle$ is c. We prove that the following statements are equivalent: (a) $TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ is isomorphic to the d-nary $(n-1)$-dimensional De Bruijn digraph; (b) α is a cyclic vector for T; (c) $TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ is primitive. This strengthens a result conjectured by C.M. Fiduccia and E.M. Jacobson [Universal multistage networks via linear permutations, in: Proceedings of the 1991 ACM/IEEE Conference on Supercomputing, ACM Press, New York, 1991, pp. 380–389]. Under the further assumption that T is invertible we show that each component of $TCP(\mathbb{Z}_d^n, \alpha, F_{T,\omega})$ is a conjunction of a cycle and a De Bruijn digraph, namely a generalized wrapped butterfly. Finally, we discuss the affine TCP digraph representations for a class of digraphs introduced by D. Coudert, A. Ferreira and S. Perennes [Isomorphisms of the De Bruijn digraph and free-space optical networks, Networks 40 (2002) 155–164].

Keywords: Affine transformation; De Bruijn digraph; Wrapped butterfly; Transformation coset pseudo-digraph

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E-mail address: ykwu@sjtu.edu.cn (Y. Wu).

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1. Introduction

This paper is about an interesting phenomenon, namely sometimes some digraphs arising from seemingly very general algebraic constructions or those restricted by a simple algebraic requirement turn out to be of a very regular pattern and have close connection with a family of seemingly very special digraphs, the so-called De Bruijn digraphs [2]. Perhaps we should call this special type of digraphs universal digraphs as they already appear in a wide range of research [1–22]. In this sense, our work here will confirm the assertion that the mysterious De Bruijn digraphs are really universal. Let us postpone a more accurate description of the phenomenon referred to above to the end of this section and review first some preliminary definitions to be used in this paper.

As usual, $\mathbb{Z}_d$ denotes the ring of integers modulo d and $\mathbb{Z}_d^n$ represents the set of $1 \times n$ matrices over $\mathbb{Z}_d$. $\mathbb{Z}_d^n$ can be viewed as the n-dimensional free module over $\mathbb{Z}_d$ and has a *standard basis* $\{e_0, e_1, \dots, e_{n-1}\}$, where e_i is the vector with a single 1 in the $(i+1)$th position and 0's elsewhere. A subset $\{\alpha_1, \dots, \alpha_t\}$ of $\mathbb{Z}_d^n$ is called *linearly independent* over $\mathbb{Z}_d^n$ if and only if whenever $\sum_{i=1}^t k_i\alpha_i = 0$ with $k_i \in \mathbb{Z}_d$, then $k_1 = k_2 = \cdots = k_t = 0$.

Let $Mat_n(\mathbb{Z}_d)$ be the set of all $n \times n$ matrices over $\mathbb{Z}_d$. The set of invertible matrices in $Mat_n(\mathbb{Z}_d)$ is denoted $GL_n(\mathbb{Z}_d)$. Arbitrarily picking $T \in Mat_n(\mathbb{Z}_d)$ and $\omega \in \mathbb{Z}_d^n$, the *affine transformation* $F_{T,\omega}$ on $\mathbb{Z}_d^n$ is defined by $F_{T,\omega}(x) = xT + \omega, \forall x \in \mathbb{Z}_d^n$. For $i = 0, 1, \dots, n-1$, let $T(i, \cdot) = e_i T$ and $T(\cdot, i) = Te_i^{\top}$, respectively.

For $S \subseteq \mathbb{Z}_d^n$, the submodule generated (or spanned) by S is $\langle S \rangle \doteq \{\sum_{i=1}^m c_i s_i : c_i \in \mathbb{Z}_d, s_i \in S, m \geq 0\}$. If a submodule M is spanned by a set S, we call S a *generating set* of M. For any nonzero vector $\alpha \in \mathbb{Z}_d^n$ and $T \in Mat_n(\mathbb{Z}_d)$, the T-*cyclic submodule* generated by α is the submodule $\mathbb{Z}_d(\alpha; T) \doteq \langle \{\alpha T^k, k \geq 0\} \rangle$. A vector α is a *cyclic vector* for T provided $\mathbb{Z}_d(\alpha; T) = \mathbb{Z}_d^n$. For any finite set S, $\#S$ denotes its cardinality.

Let Γ be a digraph. The vertex set and the arc set of Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. For a subset V_0 of $V(\Gamma)$, we write $N_\Gamma(V_0)$ for the out-neighbor set of V_0, which is $\{w \in V(\Gamma) : \exists u \in V_0, e \in E(\Gamma), e$ starts from u and ends at $w\}$. We let $N_\Gamma^0(V_0) = V_0$ and define inductively that $N_\Gamma^k(V_0) = N_\Gamma(N_\Gamma^{k-1}(V_0))$ for any positive integer k. A digraph Γ is *strongly connected* if for any two vertices x and y of Γ, there always exists in Γ a path from x to y. We say that a digraph is *connected* if its underlying undirected graph is connected. The *components* of a digraph refer to its connected components. A digraph Γ is said to be *balanced* if the in-degree and out-degree of each of its vertices are equal. We write $G \cong H$ to denote that G and H are isomorphic digraphs. If $G \cong H$, we think of them as different representations of the same object and thus often do not distinguish between them.

Given an $\alpha \in \mathbb{Z}_d^n$ and a transformation F on $\mathbb{Z}_d^n$, the *transformation coset pseudo-digraph* (TCP digraph, for short) of $\mathbb{Z}_d^n$ with respect to them, denoted $TCP(\mathbb{Z}_d^n, \alpha, F)$, is the digraph whose vertex set is $\mathbb{Z}_d^n/\langle\alpha\rangle$ and the number of arcs from vertex $x + \langle\alpha\rangle$ to vertex $\#\{z \in x + \langle\alpha\rangle : F(z) \in y + \langle\alpha\rangle\}$. Let S be a union of several cosets of $\langle\alpha\rangle$ in $\mathbb{Z}_d^n$. If there are no arcs between $S/\langle\alpha\rangle$ and $(\mathbb{Z}_d^n - S)/\langle\alpha\rangle$ in $TCP(\mathbb{Z}_d^n, \alpha, F)$, then we use the notation $TCP(S, \alpha, F)$ to represent the subdigraph induced by the vertex set $S/\langle\alpha\rangle$, which is a TCP digraph on S.

The *conjunction* $\Gamma_1 \otimes \Gamma_2$ of two digraphs Γ_1 and Γ_2 has $V(\Gamma_1) \times V(\Gamma_2)$ as the vertex set and $E(\Gamma_1 \otimes \Gamma_2)$ has $(x_1, x_2)(y_1, y_2)$ as an element of multiplicity $m_1 m_2$, where m_i