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Schur partial derivative operators

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Abstract

A lattice diagram is a finite list $L = ((p_1, q_1), \ldots, (p_n, q_n))$ of lattice cells. The corresponding lattice diagram determinant is $\Delta_L(X; Y) = \det \|x_i^{p_j} y_i^{q_j}\|$. The space M_L is the space spanned by all partial derivatives of $\Delta_L(X; Y)$. We describe here how a Schur function partial derivative operator acts on lattice diagrams with distinct cells in the positive quadrant.

1. Introduction

We consider the symmetric group $\mathcal{S}_n$ acting diagonally on $\mathbb{Q}[x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n]$, the polynomial ring in $2n$ variables. More specifically, for $\sigma \in \mathcal{S}_n$, we consider the following (diagonal) action on polynomials:

$$\sigma P(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) = P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}; y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_n}).$$

A polynomial $\Delta = \Delta(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n)$ is said to be *alternating* if, for all $\sigma \in \mathcal{S}_n$, we have $\sigma\Delta = sign(\sigma)\Delta$. It is well known that the set of all lattice diagram determinants (described in the next section) forms a basis for the space of alternating polynomials.

Given an alternating polynomial Δ we are interested in the space $\mathcal{L}_\partial[\Delta]$ spanned by all possible partial derivatives of Δ. Since the diagonal action of $\mathcal{S}_n$ commutes with applying

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partial derivatives, the space $\mathcal{L}_\partial[\Delta]$ is an $\mathcal{S}_n$-module. Our goal is to give a complete description of its (graded) character. This is a very hard problem in general and even the simplest cases require elaborate constructions [4,6,7]. The aim of the present work is to develop tools that will allow us to better achieve this goal.

In previous work [1–4] we remark that the first step in describing the structure of $\mathcal{L}_\partial[\Delta]$ is to determine its subspace of alternating polynomials. This subspace corresponds to the space spanned by all *symmetric* partial derivative operators applied to Δ. In view of this, we need to describe explicitly how the different bases of symmetric partial derivative operators act on a given lattice diagram determinant. In the work cited above, we describe the action of power sum symmetric operators and elementary symmetric operators and homogeneous symmetric operators in one set of variables. Yet the action of one of the most important bases of symmetric partial derivative operators, namely the Schur symmetric operators, was still not explicitly given. We give such a description here. Once our formula is established, we encourage the reader to revisit the previous work on the subject. For example, some results of [5] become conceptually simpler using our description and we see exactly why the multiplicity of the sign representation in a row diagram with a hole is as given in Section 4 of [5]. Our result can also be used to give a better description, in terms of partial Schur polynomials, of the vanishing ideal for the diagrams considered in [3]. Our hope is that our contribution will help in describing the generators of the vanishing ideal for the general cases, and this will be the subject of future work.

At first it seems that the description of the Schur symmetric partial derivative operators on Δ should follow directly from the expansion of Schur symmetric functions in terms of Young tableaux, but this is not quite correct. One has to be careful with the effect of signs when applying partial derivatives to lattice determinants. We thus need to re-derive this expansion from the other basis, carefully keeping track of signs. This can be done in many ways; here we chose to use the method of [10].

2. Basic definitions

The lattice cell in the $i + 1$-st row and $j + 1$-st column of the positive quadrant of the plane is denoted by (i, j). We order the set of all lattice cells using the following *lexicographic* order:

$$(p_1, q_1) < (p_2, q_2) \qquad \Longleftrightarrow \qquad q_1 < q_2 \quad \text{or} \quad [q_1 = q_2 \text{ and } p_1 < p_2]. \tag{2.1}$$

For our purpose, a *lattice diagram* is a finite list $L = ((p_1, q_1), \ldots, (p_n, q_n))$ of lattice cells such that $(p_1, q_1) \le (p_2, q_2) \le \cdots \le (p_n, q_n)$. Following the definitions and conventions of [4], the coordinates p_i and q_i of a cell (p_i, q_i) indicate the row and column positions, respectively, of the cell. For $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_k > 0$, we say that $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ is a *partition* of n if $n = \mu_1 + \cdots + \mu_k$. We associate with a partition μ the following lattice (Ferrers) diagram $((i, j) : 0 \le i \le k - 1, 0 \le j \le \mu_{i+1} - 1)$, distinct cells ordered with (2.1), and we use the symbol μ for both the partition and its associated Ferrers diagram. For example, given the partition (4, 2, 1), its Ferrers