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Elementary divisors of Specht modules

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Abstract

Let $\mathscr{H}_q(\mathfrak{S}_n)$ be the Iwahori-Hecke algebra of the symmetric group defined over the ring $\mathbb{Z}[q, q^{-1}]$. The q-Specht modules of $\mathscr{H}_q(\mathfrak{S}_n)$ come equipped with a natural bilinear form. In this paper we try to compute the elementary divisors of the Gram matrix of this form (which need not exist since $\mathbb{Z}[q, q^{-1}]$ is not a principal ideal domain). When they are defined, we give the relationship between the elementary divisors of the Specht modules $S_q(\lambda)$ and $S_q(\lambda')$, where λ' is the conjugate partition. We also compute the elementary divisors when λ is a hook partition and give examples to show that in general elementary divisors do not exist.

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1. Introduction and statement of main results

The irreducible representations of the symmetric groups and their Iwahori–Hecke algebras have been classified and constructed by James [6] and Dipper and James [2], yet simple properties of these modules, such as their dimensions, are still not known. Every irreducible representation of these algebras is constructed by quotienting out the radical of a bilinear form on a particular type of module, known as a Specht module. The bilinear forms on the Specht modules are the objects of our study.

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One way of determining the dimension of the simple modules would be to first find the elementary divisors of the Gram matrix over $\mathbb{Z}[q, q^{-1}]$ and then specialize. This would also give the dimensions of the subquotients of the Jantzen filtrations of the Specht modules over an arbitrary field; see [7]. In general, such an approach is not possible because, as Andersen has shown, Gram matrices need not be diagonalizable over $\mathbb{Z}[q, q^{-1}]$; see [1, Remark 5.11]. We also give some examples of non-diagonalizable Specht modules in Section 7.

Let $G(\lambda)$ be the Gram matrix of the Specht module $S(\lambda)$. Then the first result in this paper shows that $G(\lambda)$ is diagonalizable if and only if $G(\lambda')$ is diagonalizable, where λ' is the partition conjugate to λ. Moreover, if $G(\lambda)$ is divisibly diagonalizable (that is, $G(\lambda)$ is equivalent to a diagonal matrix $\operatorname{diag}(d_1, \ldots, d_m)$ such that d_i divides d_{i+1}, for $1 \le i < m$), then so is $G(\lambda')$. In this case we can speak of elementary divisors and we show how the elementary divisors of $G(\lambda)$ and $G(\lambda')$ determine each other. This is a q-analogue of the corresponding result for the symmetric group [8].

We next consider the elementary divisors for the hook partitions. We show that when $\lambda = (n-k, 1^k)$, for $0 \le k < n$, the Gram matrix $G(\lambda)$ is always divisibly diagonalizable over $\mathbb{Z}[q, q^{-1}]$, and we determine the elementary divisors. Again, this is a q-analogue of the corresponding result for the symmetric groups [8]; however, the proof in the Hecke algebra case is more involved and requires some interesting combinatorics.

2. The Hecke algebra and permutation modules

Fix a positive integer n and let $\mathfrak{S}_n$ be the symmetric group of degree n.

Let R be a commutative domain and let q be an invertible element in R.

The Iwahori–Hecke algebra of $\mathfrak{S}_n$ with parameter q is the unital associative algebra $\mathscr{H}$ with generators $T_1, T_2, \ldots, T_{n-1}$ and relations

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0 && \text{for } 1 \le i < n, \\ T_i T_j &= T_j T_i && \text{for } 1 \le i < j-1 < n-1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \le i < n-1. \end{aligned}$$

Let $r_i = (i, i+1)$, for $i = 1, 2, \ldots, n-1$. Then $\{r_1, r_2, \ldots, r_{n-1}\}$ generate $\mathfrak{S}_n$ (as a Coxeter group). If $w \in \mathfrak{S}_n$ then $w = r_{i_1} \cdots r_{i_k}$ for some i_j with $1 \le i_j < n$. The word $w = r_{i_1} \cdots r_{i_k}$ is reduced if k is minimal; in this case we say that w has length k and we define $\ell(w) = k$.

If $r_{i_1} \cdots r_{i_k}$ is reduced then we set $T_w = T_{i_1} \cdots T_{i_k}$. Then T_w is independent of the choice of reduced expression for w; see, for example, [10, 1.11]. Furthermore, $\mathscr{H}$ is free as an R-module with basis $\{T_w \mid w \in \mathfrak{S}_n\}$.

A composition μ of n is a sequence of non-negative integers $(\mu_1, \mu_2, \ldots)$ that sum to n. If, in addition, $\mu_1 \ge \mu_2 \ge \ldots$, then μ is a partition of n.

Let μ be a composition of n and let $\mathfrak{S}_\mu$ be the associated Young subgroup. Then $\mathscr{H}(\mathfrak{S}_\mu) = \langle T_w \mid w \in \mathfrak{S}_\mu \rangle$ is a subalgebra of $\mathscr{H}$. Given a (right) $\mathscr{H}(\mathfrak{S}_\mu)$-module V, we define the induced $\mathscr{H}$-module

$$\operatorname{Ind}_{\mathscr{H}(\mathfrak{S}_\mu)}^{\mathscr{H}}(V) = V \otimes_{\mathscr{H}(\mathfrak{S}_\mu)} \mathscr{H}.$$