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An inequality for regular near polygons

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Abstract

Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency k and intersection numbers $a_1 > 0$, $c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ. We show

$$\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}.$$

We show the following (i)–(iii) are equivalent. (i) Equality is attained above; (ii) Γ is Q-polynomial with respect to θ_1; (iii) Γ is a dual polar graph or a Hamming graph.

MSC: 05E30

Keywords: Near polygon; Distance-regular graph; Q-polynomial; Dual polar graph; Hamming graph

1. Introduction

Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$ (see Section 2 for formal definitions). Suppose the intersection numbers $a_1 > 0$ and $c_2 > 1$. It was shown by Brouwer, Cohen and Neumaier that if Γ has classical parameters $(d, q, 0, \beta)$ then Γ is a Hamming graph or a dual polar graph [2, Theorem 9.4.4]. The same conclusion was obtained by the second author under the assumption that Γ is Q-polynomial and has

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diameter $d \geq 4$ [10, Corollary 5.7]. Let $\theta_0 > \theta_1 > \cdots > \theta_d$ denote the eigenvalues of Γ. It is known that $\theta_0 = k$, where k denotes the valency of Γ. By [2, Proposition 4.4.6(i)],

$$\theta_d \geq -\frac{k}{a_1 + 1},$$

with equality if and only if Γ is a near $2d$-gon. We now state our result.

Theorem 1.1. *Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency k, and intersection numbers $a_1 > 0$, $c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ. Then*

$$\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}. \tag{1.1}$$

Moreover, the following (i)–(iii) *are equivalent.*

(i) *Equality is attained in* (1.1);

(ii) *Γ is Q-polynomial with respect to θ_1;*

(iii) *Γ is a dual polar graph or a Hamming graph.*

2. Preliminaries

In this section we review some definitions and basic concepts. See the books by Bannai and Ito [1] or Brouwer et al. [2] for more background information.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges, with vertex set X, edge set R, path-length distance function ∂ and diameter $d := \max\{\partial(x, y) \mid x, y \in X\}$. For $x \in X$ and for all integers i, set

$$\Gamma_i(x) := \{y \mid y \in X, \partial(x, y) = i\}.$$

Let k denote a nonnegative integer. We say Γ is *regular* with *valency* k whenever $|\Gamma_1(x)| = k$ for all $x \in X$. Pick an integer i $(0 \leq i \leq d)$. For $x \in X$ and for $y \in \Gamma_i(x)$, set

$$B(x, y) := \Gamma_1(x) \cap \Gamma_{i+1}(y), \tag{2.1}$$

$$A(x, y) := \Gamma_1(x) \cap \Gamma_i(y), \tag{2.2}$$

$$C(x, y) := \Gamma_1(x) \cap \Gamma_{i-1}(y). \tag{2.3}$$

The graph Γ is said to be *distance-regular* whenever for all integers i $(0 \leq i \leq d)$, and for all $x, y \in X$ with $\partial(x, y) = i$, the numbers

$$c_i := |C(x, y)|, \qquad a_i := |A(x, y)|, \qquad b_i := |B(x, y)| \tag{2.4}$$

are independent of x and y. We call the c_i, a_i, b_i the *intersection numbers* of Γ. We observe $c_0 = 0$, $a_0 = 0$, $b_d = 0$ and $c_1 = 1$. For the rest of this paper we assume Γ is distance-regular with diameter $d \geq 3$. We observe Γ is regular with valence $k = b_0$ and that [2, p. 126]

$$c_i + a_i + b_i = k \qquad (0 \leq i \leq d). \tag{2.5}$$