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The minimum Manhattan distance and minimum jump of permutations

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A B S T R A C T

Let π be a permutation of $\{1, 2, \dots, n\}$. If we identify a permutation with its graph, namely the set of n dots at positions $(i, \pi(i))$, it is natural to consider the minimum L^1 (Manhattan) distance, $d(\pi)$, between any pair of dots. The paper computes the expected value (and higher moments) of $d(\pi)$ when $n \to \infty$ and π is chosen uniformly, and settles a conjecture of Bevan, Homberger and Tenner (motivated by permutation patterns), showing that when d is fixed and $n \to \infty$, the probability that $d(\pi) \geq d + 2$ tends to e^{-d^2-d}.

The minimum jump $\mathsf{mj}(\pi)$ of π, defined by $\mathsf{mj}(\pi) = \min_{1 \leq i \leq n-1} |\pi(i+1) - \pi(i)|$, is another natural measure in this context. The paper computes the asymptotic moments of $\mathsf{mj}(\pi)$, and the asymptotic probability that $\mathsf{mj}(\pi) \geq d + 1$ for any constant d.

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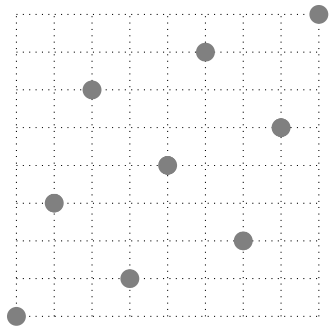

Fig. 1. The graph of the permutation $\pi = 147258369 \in \mathfrak{S}_9$.

1. Introduction

Let n be a positive integer, with $n \geq 2$. We write $[n]$ for the set $\{1, 2, \ldots, n\}$, and we write $\mathfrak{S}_n$ for the set of all permutations of $[n]$. We write a permutation $\pi \in \mathfrak{S}_n$ in one-line notation, so $\pi = \pi(1)\pi(2)\ldots\pi(n)$. Recall that the *graph* of a permutation π is the set of points (*dots*) of the form $(i, \pi(i))$ for $i \in [n]$. Fig. 1 depicts the graph of the permutation $\pi = 147258369 \in \mathfrak{S}_9$.

The *minimum Manhattan distance* $d(\pi)$ of a permutation π is defined by:

$$d(\pi) = \min_{1 \leq i < j \leq n} \{|i - j| + |\pi(i) - \pi(j)|\}. \quad (1)$$

The permutation π in Fig. 1 has $d(\pi) = 4$ (which is, in fact, the largest possible value for a permutation in $\mathfrak{S}_9$). Note that for $n \geq 2$ we have $d(\pi) \geq 2$ for all $\pi \in \mathfrak{S}_n$.

The minimum Manhattan distance is a natural measure when thinking of a permutation as its graph, but was first studied (under the name of the *breadth* of a permutation) by Bevan, Homberger, and Tenner [2] in the context of permutation patterns. We now briefly explain this context.

Two sequences $a = a_1, a_2, \ldots, a_n$ and $b = b_1, b_2, \ldots, b_n$ of distinct real numbers are said to have the same *relative order* if $a_i < a_j$ precisely when $b_i < b_j$. For a given sequence a of length n, we define the *standardisation of* a to be the unique sequence on the letters $[n]$ which is in the same relative order as a. The pattern ordering imposes a partial order on the set of all permutations: For $\pi \in \mathfrak{S}_n$ and $\sigma \in \mathfrak{S}_k$, we say that π *contains* σ *as a pattern* (denoted $\sigma \prec \pi$) if there is a subsequence of $\pi(1)\pi(2)\ldots\pi(n)$ whose standardisation is equal to $\sigma(1)\sigma(2)\ldots\sigma(k)$. For example, $213 \prec 34152$ as seen by the first, third, and fourth entries, while $21 \not\prec 123456$.

A permutation $\pi \in \mathfrak{S}_n$ can contain at most n distinct patterns of length $n{-}1$, at most $n(n{-}1)/2$ patterns of length $n{-}2$, and at most $\binom{n}{d}$ patterns of length $n{-}d$. For an integer d, a permutation is *d-prolific* if it contains precisely $\binom{n}{d}$ distinct patterns of length $n{-}d$. Equivalently, a permutation is d-prolific if every choice of d deletions yields a different pattern. The notion of a prolific permutation was introduced by Homberger [7];