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On the chromatic number of structured Cayley graphs

Mohammad Bardestani [a], Keivan Mallahi-Karai [b]

[a] DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB, United Kingdom of Great Britain and Northern Ireland
[b] Jacobs University Bremen, Campus Ring I, 28759 Bremen, Germany

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ABSTRACT

In this paper, we will study the chromatic number of a family of Cayley graphs that arise from algebraic constructions. Using Lang–Weil bound and representation theory of finite simple groups of Lie type, we will establish lower bounds on the chromatic number of a large family of these graphs. As a corollary we obtain a lower bound for the chromatic number of certain Cayley graphs associated to the ring of $n \times n$ matrices over finite fields, establishing a result for the case of SL_n parallel to a theorem of Tomon [26] for GL_n. Moreover, using Weil's bound for Kloosterman sums we will also prove an analogous result for SL_2 over certain finite rings.

1. Introduction

Let G be a group, and let S be a symmetric subset of G, that is, a set satisfying $S^{-1} = S$. Moreover assume that $\mathbf{1} \notin S$ where $\mathbf{1}$ is the identity element of G. The Cayley graph of G with respect to S, denoted by $\mathrm{Cay}(\mathrm{G}, S)$, is the graph whose vertex set is identified with G, and vertices $g_1, g_2 \in \mathrm{G}$ are declared adjacent if and only if $g_1^{-1} g_2 \in S$.

E-mail addresses: mohammad.bardestani@gmail.com (M. Bardestani), k.mallahikarai@jacobs-university.de (K. Mallahi-Karai).

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Recall also that the chromatic number of a graph $\mathcal{G}$, denoted by $\chi(\mathcal{G})$, is the least cardinal c such that the vertex set $V(\mathcal{G})$ can be partitioned into c sets (called color classes) such that no color class contains an edge in $\mathcal{G}$.

The study of chromatic number of Cayley graphs and their subgraphs was first initiated by Babai [4]. The focus of Babai's paper was on finding Cayley graphs of a given group with a *small* chromatic number. For instance, it was shown in [4] that every solvable group G has a generating set S such that the chromatic number of $\mathrm{Cay}(\mathrm{G}, S)$ is at most 3. It is easy to see ([4], Proposition 4.6) that $\chi(\mathrm{Cay}(\mathrm{G}, S)) = 2$ for some generating set S if and only if G has a subgroup of index 2.

Bounding the chromatic number of Cayley graphs from below is a more subtle problem. Alon [2] considered random Cayley graphs of arbitrary finite groups and established strong asymptotically almost sure lower bounds for their chromatic number. In the random model considered in this paper, S is a randomly chosen subset of G of a given cardinality k. Alon then establishes various lower bounds for the chromatic number of $\mathrm{Cay}(\mathrm{G}, S)$ that hold with probability converging to 1 as $n \to \infty$. In order for the bounds to be non-trivial, one needs $k \gg \log n$. In the opposite direction, Alon also proved that if G is abelian, and $k \ll \log\log n$, then with probability tending to 1 as $n \to \infty$, the inequality $\chi(\mathrm{Cay}(\mathrm{G}, S)) \leq 3$ also holds.

In this paper we will address similar problems in the case that the pair (G, S) arises from an algebraic construction, and can thus be viewed as highly structured. More precisely, let $\mathbf{G} \subseteq \mathrm{GL}_n$ be a Chevalley group. Such groups are naturally obtained from a simple complex Lie algebra [24]. $\mathbf{G}$ can also be viewed as a group scheme of finite type defined over $\mathbb{Z}$, which implies that there is a finite set $\{f_i\}_{i \in I}$ of polynomials with integer coefficients in variables x_{ij} such that for any unital ring R, the common solutions of $\{f_i\}_{i \in I}$ form a group, which is denoted by $\mathbf{G}(R)$.

The reader interested in concrete examples may consider the special case $\mathbf{G} = \mathrm{SL}_n$, which is defined by the equation $\det(x_{ij}) - 1 = 0$. Note that the set of zeros of this polynomial over any unital ring R defines the group $\mathrm{SL}_n(R)$, consisting of unimodular n by n matrices with entries in R. Let also $\tilde{\mathbf{S}}$ be an affine subscheme of $\mathbf{G}$ of finite type over $\mathbb{Z}$, namely

$$\tilde{\mathbf{S}} := \{(x_{ij}) \in \mathbf{G} : P_1(x_{ij}) = \cdots = P_r(x_{ij}) = 0\}, \tag{1}$$

where $P_1, \ldots, P_r$ are polynomials with integer coefficients in x_{ij}. Since $\mathbf{G}$ and $\tilde{\mathbf{S}}$ are defined over $\mathbb{Z}$, for any prime power q we can consider the $\mathbb{F}_q$-points of $\mathbf{G}$ and $\tilde{\mathbf{S}}$ denoted by $\mathbf{G}(\mathbb{F}_q)$ and $\tilde{\mathbf{S}}(\mathbb{F}_q)$. Here $\mathbb{F}_q$ denotes the finite field with q elements. We always assume that $\mathbf{1} \notin \tilde{\mathbf{S}}$ and denote by

$$\mathscr{G}_{\mathbf{G},\mathbf{S}}(\mathbb{F}_q) := \mathrm{Cay}(\mathbf{G}(\mathbb{F}_q), \mathbf{S}(\mathbb{F}_q)), \qquad \mathbf{S}(\mathbb{F}_q) := \tilde{\mathbf{S}}(\mathbb{F}_q) \cup \tilde{\mathbf{S}}(\mathbb{F}_q)^{-1}, \tag{2}$$

the Cayley graph of the group $\mathbf{G}(\mathbb{F}_q)$ with respect to the symmetrized set $\mathbf{S}(\mathbb{F}_q)$.